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Analytical derivation of at-a-station hydraulic–geometry relations

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Summary This paper uses generalized expressions for both cross-section geometry and hydraulics (generalization of the Chézy and Manning relations) to derive explicit equations for the exponents and coefficients in the power-law at-a-station hydraulic–geometry relations. The exponents are shown to depend only on the depth exponent in the hydraulic relation (***p***) and the exponent that reflects the form of the cross-section (***r***). The coefficients depend on ***p*** and ***r***, but also on the slope exponent in the generalized hydraulic relation and on the physical characteristics of the section: bankfull width, bankfull maximum depth, hydraulic conductance, and slope. The theoretical ranges of coefficient and exponent values derived herein are generally consistent with the averages and individual observed values reported in previous studies. However, observed values of the exponents at particular cross-sections commonly fall outside the theoretical ranges. In particular, the observed value of the velocity exponent ***m*** is commonly greater than the theoretical value, suggesting that hydraulic conductance often increases more strongly with discharge than predicted by the assumed hydraulic relations. The developments presented here provide new theoretical insight into the ways in which hydraulic and geometric factors determine hydraulic geometry. This insight should help to explain the variation of at-a-station hydraulic geometry and may facilitate prediction of hydraulic geometry at river reaches where detailed measurements are unavailable.

Introduction

The concept of hydraulic geometry was introduced by Leopold and Maddock (1953). The basic at-a-station hydraulic–geometry relations are functions relating the water-surface width, *W*, average depth, *Y*, and average velocity, *U*, to discharge, *Q*, at a particular stream cross-section or reach. The functions are usually given in the form of power-law equations:

$$W = a \cdot Q^{b}, \tag{1W}$$

$$Y = c \cdot Q^{f}, \tag{1Y}$$

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Nomenclature

[1] *denotes dimensionless quantities*

a	coefficient in width–discharge relation $[L^{1-3\cdot b}\, T^{b}]$
b	exponent in width–discharge relation [1]
c	coefficient in depth–discharge relation $[L^{1-3\cdot f}\, T^{f}]$
C	Chézy's conductance coefficient [1]
f	exponent in depth–discharge relation [1]
g	gravitational acceleration $[L\, T^{-2}]$
h	exponent in power-law velocity profile [1]
k	coefficient in velocity–discharge relation $[L^{1-3\cdot m}]$
K	generalized conductance coefficient $[L^{1-p}\, T^{-1}]$
m	exponent in velocity–discharge relation [1]
n	Manning's resistance coefficient [1]
p	depth exponent in generalized hydraulic relation [1]
q	slope exponent in generalized hydraulic relation [1]
Q	discharge $[L^3\, T^{-1}]$
r	exponent in cross-section geometry relation [1]
R	hydraulic radius [L]
S	energy slope [1]
u_C	unit-conversion factor in Chézy relation $[L^{1/2}\, T^{-1}]$
u_M	unit-conversion factor in Manning relation $[L^{1/3}\, T^{-1}]$
U	average cross-sectional velocity $[L\, T^{-1}]$
W	water-surface width [L]
W^*	bankfull water-surface width [L]
x	cross-channel distance from channel center [L]
y_r	effective height of bottom roughness elements [L]
Y	cross-sectional average water depth [L]
Y^*	bankfull cross-sectional average water depth [L]
Y_m	maximum water depth in cross-section [L]
Y_m^*	bankfull maximum water depth in cross-section [L]
z	vertical distance of channel bottom above lowest (central) point [L]
δ	$\equiv 1 + r + r\cdot p$ [1]

$$U = k \cdot Q^m. \tag{1U}$$

At-a-station hydraulic–geometry relations are useful tools in many types of hydrological analysis. They can be used directly in flood routing (Western et al., 1997; Orlandini and Rosso, 1998), and can be combined with flow-duration curves to produce water-resources-index duration curves that are useful in riverine-habitat analysis (Jowett, 1998), water-quality management, reservoir-sedimentation studies, and in determining the frequency of sediment movement (Dingman, 2002). Width–discharge relations can be used to estimate discharge via remote sensing (Bjerklie et al., 2005a).

Given the power-law forms of Eq. (1), the continuity relation,

$$Q = W \cdot Y \cdot U, \tag{2}$$

dictates that

$$a \cdot c \cdot k = 1 \tag{3}$$

and

$$b + f + m = 1. \tag{4}$$

It is important to note that at-a-station hydraulic–geometry relations as commonly applied are valid only for *in-bank* flows. Garbrecht (1990) expanded the concept by showing that two empirical power functions could be connected to apply to in-bank and over-bank flows at a given section. However, the discussion here is limited to in-bank flows.

Eq. (1) each plot as straight lines on double-logarithmic graph paper. The values of the coefficients (antilogs of the intercepts) and exponents (slopes) of these relations at a given cross-section are usually determined empirically by ordinary least-squares regression analysis on the logarithms of values of W, Y, U, and Q collected during discharge measurements at the section.

Ferguson (1986) reviewed empirical studies of at-a-station hydraulic geometry and theoretical attempts to explain why the exponents tend toward particular central values (though with much variability) and relative values (e.g., $b < f$). His main conclusion was that, given a cross-section with a specified constant shape and frictional characteristics and a law relating average velocity to friction and depth, the within-bank at-a-station hydraulic–geometry relations are determined. Thus he rejected theoretical approaches to determining the exponents in Eqs. (1W)–(1U) that invoke a ''metaphysical'' explanation, such as an assumed tendency toward ''minimum variance'' (Langbein, 1964).

Ferguson (1986) also showed that the $W(Q)$, $Y(Q)$, and $U(Q)$ relations will be power-laws only if the $W(Y)$ and $U(Y)$ relations are power-laws, such as the commonly used Manning and Chézy relations:

$$\text{Manning}: U = \frac{u_M}{n} \cdot Y^{2/3} \cdot S^{1/2}; \tag{5}$$

$$\text{Chézy}: U = u_C \cdot C \cdot Y^{1/2} \cdot S^{1/2}, \tag{6}$$

where u_M and u_C are unit-conversion factors (for SI units $u_M = 1$, $u_C = 0.552$; for British units $u_M = 1.49$, $u_C = 1$), n is Manning's resistance coefficient, S is water-surface slope, C is Chézy's conductance coefficient, and Y is assumed to differ negligibly from the hydraulic radius, R (i.e., $W/Y > 10$). Applying the Manning equation to a parabolic channel, a form commonly assumed to be approximated by natural river cross-sections, Ferguson (1986) found that $b = 0.23$, $f = 0.46$, and $m = 0.31$, ''strikingly close'' to the central values of empirical values reported in the literature. He also noted that almost no attention has been given to the factors that determine the coefficients a, c, and k.

Ferguson's (1986) conclusion that at-a-station hydraulic geometry is completely determined by cross-section geom-