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Direct propagation of probability density functions in hydrological equations

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Abstract

Sustainable decisions in hydrological risk management require detailed information on the probability density function (*pdf*) of the model output. Only then probabilities for the failure of a specific management option or the exceedance of critical thresholds (e.g. of pollutants) can be derived. A new approach of uncertainty propagation in hydrological equations is developed that directly propagates the probability density functions of uncertain model input parameters into the corresponding probability density functions of model output. The basics of the methodology are presented and central applications to different disciplines in hydrology are shown. This work focuses on the following basic hydrological equations: (1) pumping test analysis (Theis-equation, propagation of uncertainties in recharge and transmissivity), (2) 1-dim groundwater contaminant transport equation (Gauss-equation, propagation of uncertainties in decay constant and dispersivity), (3) evapotranspiration estimation (Penman–Monteith-equation, propagation of uncertainty in roughness length). The direct propagation of probability densities is restricted to functions that are monotonically increasing or decreasing or that can be separated in corresponding monotonic branches so that inverse functions can be derived. In case no analytic solutions for inverse functions could be derived, semi-analytical approximations were used. It is shown that the results of direct probability density function propagation are in perfect agreement with results obtained from corresponding Monte Carlo derived frequency distributions. Direct *pdf* propagation, however, has the advantage that is yields exact solutions for the resulting hydrological *pdfs* rather than approximating discontinuous frequency distributions. It is additionally shown that the *type* of the resulting *pdf* depends on the specific values (order of magnitude, respectively) of the standard deviation of the input *pdf*. The dependency of skewness and kurtosis of the propagated *pdf* on the coefficient of variation of input parameter uncertainty is detected to be non-monotonic with distinctive maxima.

Keywords: Uncertainty propagation; Probability density function; Pumping test analysis; Theis-equation; 1-dim groundwater contaminant transport equation; Penman-Monteith-equation; Monte Carlo simulation

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1. Background and motivation

While deterministic hydrological modelling assumes that all parameters appearing in the model equations are known perfectly, stochastic modelling accounts for the fact that hydrological properties vary spatially and are in particular not known perfectly. In reality, this is always the case for field experiments but also for laboratory experiments. The imperfect knowledge of model input parameters in turn affects the reliability of model output. This circumstance is of crucial importance when model results are used for decision-making. The quantification of uncertainty ranges of model output is a prerequisite for decisions that aim at sustainable management of water resources. It is crucial to consider deviations from the model mean response.

Monte Carlo techniques were applied very successfully in the last decades in hydrological sciences and allowed to quantify the model output range under a given parameter input distribution. An extension of the Monte Carlo assessment is the application of the Metropolis algorithm (e.g. Kuczera and Parent (1998), which uses a random walk that adapts to the true input parameter *pdf*. Vrugt et al. (2003) apply the *Suffled Complex Evolution Metropolis* algorithm (SCEM-UA) to derive posterior hydrologic model parameter distributions and model output uncertainty bounds. Yu et al. (2001) examined the uncertainty of a simple rainfall-runoff model output caused by model parameters via four different methods: beside the application of Rosenblueth's point estimation (Rosenblueth, 1981) and Harr's point estimation (Harr, 1989), also classical Monte Carlo Simulation and Latin Hypercube Sampling were applied to build uncertainty bounds on an estimated hydrograph. A further popular extension of the Monte Carlo method is GLUE (*Generalised Likelihood Uncertainty Estimation*) (e.g. Beven and Freer, 2001). GLUE rejects the concept of an optimum model and parameter set and assumes that all parameter sets have an equal likelihood of being acceptable. Realizations that are deemed unacceptable or non-behavioural are rejected by being given a likelihood of zero. Wagener et al. (2003) introduce the Dynamic Identifiability Analysis (DYNIA) for quantification of uncertainties in conceptual rainfall-runoff models. Krzysztofowicz (2002) proposed Bayesian techniques for the estimation of uncertainty in rainfall-runoff models (*Bayesian Forecasting System*, BFS). For many applications in hydrological risk assessment also fuzzy set techniques were applied successfully (e.g. Bardossy and Duckstein, 1995).

Due to the computational demand of Monte Carlo based methods for uncertainty quantification, moment propagation techniques were developed that are able to directly calculate first and second moments without application of Monte Carlo simulation (e.g. Protopapas and Bras, 1990; Kunstmann et al., 2002). Many decisions in hydrological risk management, however, require detailed information on the entire probability density function (*pdf*) of model output. With respect to prediction of extreme events, in particular the tails of distributions are of importance. Only if probability distributions are known, probabilities for the failure of a specific management option like, e.g. the exceedance of critical pollutant concentrations can be derived. Monte Carlo simulations can satisfy this need but they tend to be computationally demanding. Moreover, they provide discontinuous histograms and not a continuous *pdf*. To improve the understanding of uncertainty propagation we investigated the principles of a new approach that directly propagates the probability density functions of uncertain model input parameters into the corresponding probability density functions of model output. This paper presents the basic principles of the methodology and shows basic applications to different fields in hydrology.

2. Theory

The probability density function $p(x)$ of a continuous random variable x allows to calculate the probability that the random variable x is within the interval $[x_1, x_2]$:

$$P(x_1 < x \leq x_2) = \int_{x_1}^{x_2} p(y)\mathrm{d}y. \tag{1}$$

Every *pdf* fulfils the condition $\int_{-\infty}^{-\infty} p(y)\mathrm{d}y = 1$.