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Fitting the log-logistic distribution by generalized moments

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Summary The method of generalized moments (GM) is investigated for parameter and quantile estimation in the 2-parameter log-logistic (LL2) model. Point estimators for the shape and scale parameters and quantiles are derived. Asymptotic variances and covariances for these estimators are presented, along with simulation results on the performance of the GM method versus the methods of generalized probability weighted moments (GPWM), of maximum likelihood (ML), and of classical moments applied to $Y = \ln X$. The GPWM and ML methods have already been investigated by the authors. Some mathematical properties of the LL2 model and some relationships between GM and GPWM are highlighted. The simulation results show the GM method to outperform the other competitive methods in the LL2 case, when moment orders are appropriately chosen. It is also shown that a mixture of moments of positive and negative orders is needed for optimal estimation under an LL2 model, and how this mixture can be implemented using the GM method. However, further research into the area of optimal choice of moment orders is still needed. Mixing positive and negative moments in the estimation is demonstrated by a hydrological example involving low stream flow.

Introduction

Statistical frequency analysis using probability distributions is widely employed in hydrology for estimating the relation between the magnitude and occurrence frequency of various hydrological events. A procedure commonly used involves: (i) selecting a sample of values of the hydrological variable that satisfies the criteria of randomness, independence, homogeneity and stationarity; (ii) fitting a probability distribution to this sample by an appropriate fitting method, and (iii) using this fitted distribution to make statistical inferences about the underlying population.

Following the publication of the Flood Estimation Handbook (FEH) in the UK (IH, 1999) and the recommendation of the generalized logistic (GL) distribution as the standard

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for flood frequency analysis in that country, the use of this distribution has increased in popularity in hydrology. The GL distribution, as used in the FEH, is in fact a generalization of the 3-parameter log-logistic (LL3) distribution, which had earlier been examined by Ahmad et al., 1988) in an at-site and regional study involving Scottish flood data. Ahmad et al., 1988) compared the LL3 distribution to the GEV, the 3-parameter lognormal, and the Pearson type 3 models, and found that the LL3 model performed ''extremely well'' compared to these other models, according to a set of criteria chosen by the authors. For the exact relationship between the GL, LL3 and logistic distributions, the reader may refer to the two studies: Kjeldsen and Jones (2004), and Ahmad et al., 1988). From these two studies, it can be deduced that the LL3 distribution of Ahmad et al., 1988) is a special case of the GL distribution (corresponding to the case $k < 0$, according to the notation by Kjeldsen and Jones (2004)).

On the other hand, Shoukri et al., 1988) showed a good fit of the 2-parameter LL model (LL2) to precipitation data from various Canadian regions. More recently, Ashkar and Mahdi (2003) also compared the LL2 model to the 2-parameter lognormal, the 2-parameter Weibull, and the extreme value type 1 distributions for fitting maximum annual stream flow data. This comparison showed the good fitting potential of the LL2 distribution to a large data set (114 hydrometric series). In a separate study, various 2-parameter distributions were also considered by Ashkar et al., 2004) for fitting low stream-flow data by the deficit-below-threshold (DBT) approach, as will be described in ''Hydrological example'' of the present study. Following that study, in which seven types of distributions were considered, the LL2 distribution was one of the models recommended for fitting low-flow volume, intensity and duration.

Among the methods used to fit statistical distributions to hydrological data, the maximum likelihood (ML) method has long been considered important, due to its asymptotic efficiency. On the other hand, the method of moments (MM) has been popular due to its ease of application, and the method of probability-weighted moments (PWM) (Greenwood et al., 1979; Hosking et al., 1985; Hosking, 1986) has been quite widely applied as an alternative to the MM and ML methods.

In 2001, Rasmussen proposed a ''generalization'' of the PWM method, which he called method of generalized probability weighted moments (GPWM), and applied this method to the 2-parameter Pareto distribution (although according to a referee, this ''generalized'' PWM method was the one originally proposed by Greenwood et al., 1979). Ashkar and Mahdi (2003) developed the GPWM for the LL2 distribution, and compared it to the ML method.

In the present study, we will develop the method of generalized moments (GM) for parameter and quantile estimation under an LL2 model. Point estimators will be derived and asymptotic variances and covariances of these estimators will be presented. Simulations will also be performed to compare the GM, GPWM and ML methods, along with the method of ''log-moments'' (LM), which is the classical method of moments applied to the logistic random variable $Y = \ln X$. Special attention will be paid to the flexibility of moment orders that can (and should) be used to fit statistical distributions to observed data. We will also make some comparisons between the GM and GPWM approaches to parameter estimation and, along the way, point out some interesting results concerning the LL2 distribution.

We will organize the paper as follows. In ''GM versus GPWM'', we will make some comparisons between the GM and GPWM methods for parameter estimation and in ''Generalities concerning LL2 and estimation using GM'' will give some generalities concerning the LL2 distribution and develop the necessary theory for GM estimation under this model. ''Comparison of the GM, GPWM and ML and LM methods'' will present some simulation results that compare the GM, GPWM, ML and LM methods, and ''Hydrological example'' will contain a hydrological example. Finally, ''Conclusion'' will be devoted to some concluding remarks. We will present the asymptotic variances and covariances of parameter and quantile estimators obtained by the GM method under an LL2 model in Appendix.

GM versus GPWM

In the next section, we will mathematically develop the GM method specifically for the LL2 distribution. However, we will start in the present section by making some basic reflections on the GM and GPWM methods, as applied to *any* distribution.

To estimate the parameters of a distribution, the studies by Rasmussen (2001), and by Ashkar and Mahdi (2003), used PWMs of the form

$$M_{l,r} = E[X^l F^r] = \int_{-\infty}^{\infty} x^l F^r(x) f(x)\mathrm{d}x, \tag{1}$$

where X is the hydrological continuous variable whose distribution is being estimated and F is the cumulative distribution function (cdf) of X, (Rasmussen's study also considered PWMs of the form $E[X^l(1-F)^r]$). The PWM and GPWM methods are computationally practical when the inverse of F can be calculated analytically, and the integral in (1) can be evaluated to yield an analytical expression of PWMs, as functions of distribution parameters. However, it is worth noting that, once $M_{l,r}$ in (1) is calculated, it serves as a direct basis for applying not only the GPWM method, but also the GM method, as it will clearly be seen in the following section. The GM method has been earlier used in hydrology, for instance, by Ashkar and Bobée (1987).

The idea behind the PWM and GPWM methods is to obtain parameter estimates by equating PWMs ($M_{l,r}$), to sample PWM estimates $(\hat{M}_{l,r})$, and solving the resulting system of equations for the distribution parameters. In the ''traditional'' PWM method, r values are chosen to be nonnegative integers that are as small as possible, so for a 2-parameter distribution, this method involves consideration of $r = 0$ and $r = 1$ in (1) (with $l = 1$). In the ''GPWM'' method on the other hand (as called by Rasmussen, 2001), r has neither to be small, nor a nonnegative integer.

PWM and GPWM applications have restricted attention to the case where $l = 1$ in Eq. (1), i.e., they have used only PWMs of the form $M_{l,r;l=1}$ in the estimation. The justification often provided for this restriction, is that using moments of X, that are of order greater than 1, ''should be avoided''. On the other hand, in the classical method of (product) moments, as well as in the method of generalized (product) moments (GM), which is the subject of the present paper,