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Complex phase space and Weyl's commutation relations

Sergio Albeverio[a], Ambar N. Sengupta[b,*]

[a] Institut für Angwewandte Mathematik and HCM, Universität Bonn, Bonn, Germany
[b] Department of Mathematics, Louisiana State University, United States

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Abstract

We revisit von Neumann's determination of the representations of the canonical commutation relations in Weyl form. We present an exposition of the original insights set within the convenient notational framework of symplectic structures. We study von Neumann's projection operator and show how the complex phase space representation arises.

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1. Introduction

The purpose of this paper is to revisit von Neumann's proof [29] of the equivalence of any representation of the Heisenberg commutation relations, in Weyl's exponentiated form, to a direct sum of the Schrödinger representation. We also show that von Neumann's method leads naturally to the representation of the commutation relations on a space of holomorphic functions on complex phase space.

We begin with a summary of the principal results.

* Corresponding author.
E-mail addresses: albeverio@uni-bonn.de (S. Albeverio), ambarnsg@gmail.com (A.N. Sengupta).

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We work with a finite-dimensional real vector space X with symplectic structure ω, and a complex Hilbert space $\mathbb{H}$. By a *Weyl projective representation* of (X, ω) we mean a mapping W that associates to each $x \in X$ a unitary operator $W(x)$ on $\mathbb{H}$ that depends strongly continuously on x in the sense that

$$X \to \mathbb{H} : x \mapsto W(x)v$$

is continuous for every $v \in \mathbb{H}$, and for which

$$W(x)W(y) = e^{\frac{i}{2}\omega(x,y)}W(x+y) \tag{1.1}$$

for all $x, y \in X$. We say that W is *irreducible* if $\mathbb{H} \neq \{0\}$ and $\mathbb{H}$ is the only nonzero closed subspace mapped into itself by $W(x)$ for every $x \in X$. The scalar multiplier on the right hand side of (1.1) disappears when we consider the action of $W(x)$ on the projective space $P^1(\mathbb{H})$ whose points are the one dimensional subspaces of $\mathbb{H}$; thus W specifies a genuine action of the additive group X on $P^1(\mathbb{H})$.

For many basic physical systems the phase space can be coordinatized in terms of a configuration space L and a corresponding momentum space which is the dual space L^*. Thus suppose that we have a linear mapping

$$X \to L^* \oplus L \tag{1.2}$$

that makes the symplectic space (X, ω) isomorphic to $L^* \oplus L$ with its standard symplectic structure ω_L given by

$$\omega_{\mathrm{L}}\big((p,q),(p',q')\big) = \langle p', q\rangle - \langle p, q'\rangle, \tag{1.3}$$

for all $p, p' \in L^*$ and $q, q' \in L$, with $\langle \cdot, \cdot \rangle$ being the pairing of L^* and L. Using the isomorphism (1.2) and a Weyl projective representation W of (X, ω), we define operators

$$U(q) = W(0, q) \quad \text{and} \quad V(p) = W(p, 0)$$

for all $q \in L$ and $p \in L^*$; then from the condition (1.1) we can verify that these operators satisfy the Weyl commutation relations

$$\begin{aligned} U(q)V(p) &= e^{i\langle p,q\rangle}V(p)U(q) \\ U(q)U(q') &= U(q+q') \\ V(q)V(q') &= V(q+q') \end{aligned} \tag{1.4}$$

for all $p \in L^*$ and $q \in L$. Conversely, strongly continuous unitary-operator valued functions U on L and V on L^*, satisfying the relation (1.4), specify a Weyl projective representation W specified by

$$W(p,q) = e^{-\frac{i}{2}\langle p,q\rangle}U(q)V(p). \tag{1.5}$$

This approach, introduced by Weyl [30], provides an enormously fruitful mathematically precise formulation of the commutation relations (2.1) of quantum mechanics.

Following von Neumann [29] we prove several results in Section 4. We summarize the main results here. The first result demonstrates a kind of semisimplicity of representations of the type W.