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An isoperimetric constant for signed graphs

Antoine Gournay

T.U. Dresden, Fachrichtung Mathematik, Institut für Geometrie, 01062 Dresden, Germany

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Abstract

A sign in the usual Laplacian on graphs is introduced and the corresponding analogue of the isoperimetric constant for this Laplacian is presented, *i.e.* a geometric quantity which enables to bound from above and below the first eigenvalue. The introduction of the sign in the Laplacian is motivated by the study of 2-lifts of graphs and of the combinatorial Laplacian in higher degree.

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1. Introduction

1.1. Signed graphs and twisted Laplacian

Take a graph and assign to each edge e a sign $\mathrm{sgn}\, e \in \{-1, 1\}$. The twisted Laplacian with respect to this signing is the operator $\Delta^\tau : \mathbb{R}^V \to \mathbb{R}^V$ defined by

$$\Delta^\tau f(x) = |N(x)| f(x) - \sum_{y \in N(x)} \mathrm{sgn}(x, y) f(y)$$

where $N(x)$ is the set of vertices neighbouring x (and $|N(x)|$ is the valency of x).

E-mail address: antoine.gournay@mailbox.tu-dresden.de.

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Note that putting a negative sign on all edges gives the "signless Laplacian", see, for example, the book by Cvetković, Rowlinson & Simić [5, Section 7.8].

To each cycle C, one may assign $\mathrm{sgn} C = \prod_{e \in C} \mathrm{sgn} e$. A coherent signing (or a coherently signed graph, also called a balanced graph) is a signing such that every cycle has positive sign, *i.e.* there is an even number of negatively signed edges along each cycle. If all edges have $\mathrm{sgn} e = -1$, then the graph is coherently signed if and only if there are no odd cycles (*i.e.* if and only if the graph is bipartite).

Given $S \subset V$, denote ∂S to be the set of edges between S and S^{c}. Let $e_{mc}(S)$ to be the minimal number of edges that need to be removed so that the graph induced on S is coherently signed. Let

$$\psi(S) = \frac{|\partial S| + 2e_{mc}(S)}{|S|} \quad \text{and} \quad \psi(G) = \min_{\varnothing \neq S \subset G} \psi(S).$$

One should probably stress that, as opposed to the isoperimetric constant in the classical (non-signed) case, there is no upper bound on the size of S (*e.g.* $|S| \leq |V|/2$). This comes from the fact that the twisted Laplacian has "generically" no kernel, *i.e.* the smallest *possibly zero* eigenvalue is the eigenvalue of interest. The aim of this note is to show that $\psi(G)$ is the correct analogue to the isoperimetric constant in this situation, give a sharper estimate than those given previously in the literature and to put forth the link with the Laplacian in higher degree.

The author's main sources of interest to study such operators come from 2-lifts and higher-order Laplacian. The main result of this paper is to tighten a lower bound on the spectrum of Δ^τ and shows the connection to the Laplacian in higher degree.

It is well-known (see Proposition 2.2) that this constant is 0 if and only if the graph has a coherent signing. Note that in the classical case all signs are positive, hence the signing is always coherent. This is to be expected since $\psi(G)$ bounds above and below the smallest (possibly 0!) eigenvalue of the twisted Laplacian. Some interpretation of ψ can be found at the beginning of Section 2.3.

The first appearance of the above quantity dates probably back to Desai & Rao [7], who introduced it for graphs where the signs are all negative. This operator has already been introduced and studied, see for example Belardo [3] and references therein. The referee pointed out to the author that a slighter weaker constant is obtained by Hou, see [8]. It is also possible to study generalisations (known as connection Laplacians) where the sign is replaced by an element of the orthogonal matrices $\mathrm{O}_d(\mathbb{R})$, see Bandeira, Singer & Spielman [2]. The study of n-lifts has also been gaining interest and, in that case, the signing should be an element of the symmetric group on n elements.

1.2. Results

A first (and straightforward) result (see Section 2) is that the spectrum of the twisted Laplacian lies in $\mathbb{R}_{\geq 0}$. Some bounds are more naturally expressed in terms of a quantity with an almost cosmetic difference to ψ: $\tilde{\psi}(G) = \min_{\varnothing \neq S \subset G} \frac{|\partial S| + 4e_{mc}(S)}{|S|}$.