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Approximate homomorphisms and derivations in proper JCQ^*-triples via a fixed point method

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Abstract

In this paper, we investigate homomorphisms and derivations in proper JCQ^*-triples with the following functional equation:

$$\frac{1}{k} f(kx + ky + kz) = f(x) + f(y) + f(z)$$

for a fixed positive integer k. We moreover prove the generalized Hyers–Ulam stability of homomorphisms in proper JCQ^*-triples and of derivations on proper JCQ^*-triples via a fixed point method.

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1. Introduction and preliminaries

As is extensively discussed in [17], the full description of a physical system $\mathcal{S}$ implies the knowledge of three basic ingredients: the set of the observables, the set of the states

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and the dynamics that describes the time evolution of the system by means of the time dependence of the expectation value of a given observable on a given state. Originally the set of the observables was considered to be a C^*-algebra [10]. In many applications, however, this was shown not to be the most convenient choice and the C^*-algebra was replaced by a von Neumann algebra, because the role of the representation turns out to be crucial mainly when long range interactions are involved (see [2] and references therein). Here we use a different algebraic structure, similar to the one considered in [8], which is suggested by the considerations above: because of the relevance of the unbounded operators in the description of $\mathcal{S}$, we will assume that the observables of the system belong to a quasi-$*$-algebra (A, A_0) (see [18] and references therein), while, in order to have a richer mathematical structure, we will use a slightly different algebraic structure: (A, A_0) will be assumed to be a proper CQ^*-algebra, which has nicer topological properties. In particular, for instance, A_0 is a C^*-algebra.

Let A be a linear space and A_0 be a $*$-algebra contained in A as a subspace. We say that A is a quasi-$*$-algebra over A_0 if

(i) the right and left multiplications of an element of A and an element of A_0 are defined and linear;
(ii) $x_1(x_2a) = (x_1x_2)a$, $(ax_1)x_2 = a(x_1x_2)$ and $x_1(ax_2) = (x_1a)x_2$ for all $x_1, x_2 \in A_0$ and all $a \in A$;
(iii) an involution $*$, which extends the involution of A_0, is defined in A with the property $(ab)^* = b^*a^*$ whenever the multiplication is defined.

A quasi-$*$-algebra (A, A_0) is said to be a locally convex quasi-$*$-algebra if in A a locally convex topology τ is defined such that

(i) the involution is continuous and the multiplications are separately continuous;
(ii) A_0 is dense in $A[\tau]$.

Throughout this paper, we suppose that a locally convex quasi-$*$-algebra $(A[\tau], A_0)$ is complete. For an overview on partial $*$-algebra and related topics we refer the reader to [1].

In a series of papers [3,4], many authors have considered a special class of quasi-$*$-algebras, called *proper CQ^*-algebras*, which arise as completions of C^*-algebras. They can be introduced in the following way:

Let A be a right Banach module over the C^*-algebra A_0 with involution $*$ and C^*-norm $\|\cdot\|_0$ such that $A_0 \subset A$. We say that (A, A_0) is a *proper CQ^*-algebra* if

(i) A_0 is dense in A with respect to its norm $\|\cdot\|$;
(ii) $(ab)^* = b^*a^*$ whenever the multiplication is defined;
(iii) $\|y\|_0 = \sup_{a\in A, \|a\|\le 1} \|ay\|$ for all $y \in A_0$.

Ulam [19] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

We are given a group G and a metric group G' with metric $\rho(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then a homomorphism $h : G \to G'$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G$?

By now an affirmative answer has been given in several cases, and some interesting variations of the problem have also been investigated.