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Expo. Math. 33 (2015) 431–451

EXPOSITIONES MATHEMATICAE

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Remarks on the dynamics of the horocycle flow for homogeneous foliations by hyperbolic surfaces

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Received 3 November 2013; received in revised form 13 April 2015

Dedicated to Pierre Molino with admiration

Abstract

This article is a first step towards the understanding of the dynamics of the horocycle flow on foliated manifolds by hyperbolic surfaces. This is motivated by a question formulated by M. Martínez and A. Verjovsky on the minimality of this flow assuming that the "natural" affine foliation is minimal too. We have tried to offer a simple presentation, which allows us to update and shed light on the classical theorem proved by G.A. Hedlund in 1936 on the minimality of the horocycle flow on compact hyperbolic surfaces. Firstly, we extend this result to the product of $PSL(2, \mathbb{R})$ and a Lie group G, which places us within the homogeneous framework investigated by M. Ratner. Since our purpose is to deal with non-homogeneous situations, we do not use Ratner's famous Orbit-Closure Theorem, but we give an elementary proof. We show that this special situation arises for homogeneous Riemannian and Lie foliations, reintroducing the foliation point of view. Examples and counter-examples take an important place in our work, in particular, the very instructive case of the solvable manifold T^3_A. Our aim in writing this text is to offer to the reader an accessible introduction to a subject that was intensively studied in the algebraic setting, although there still are unsolved geometric problems.

MSC 2010: primary 37D40; secondary 37C85; 57R30

Keywords: Horocyclic flow; Lie group; Minimal foliation

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http://dx.doi.org/10.1016/j.exmath.2015.07.006
0723-0869/

1. Introduction and motivation

In this paper, we start by focusing our attention on the following subgroups

$$U = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \Big/ \, t \in \mathbb{R} \right\} \quad \text{and} \quad B = \left\{ \begin{pmatrix} \lambda & t \\ 0 & \lambda^{-1} \end{pmatrix} \Big/ \, t \in \mathbb{R}, \lambda \in \mathbb{R}_*^+ \right\}$$

of the group $PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\{\pm Id\}$. We also consider a connected Lie group G and the natural right actions of U and B on the product $PSL(2,\mathbb{R}) \times G$ where every element of $PSL(2,\mathbb{R})$ acts trivially on the second factor G. We discuss the minimality of the right actions of U and B induced on the left quotient $X = \Gamma \backslash PSL(2,\mathbb{R}) \times G$ by a cocompact discrete subgroup of $PSL(2,\mathbb{R}) \times G$. Recall that an action is said to be *minimal* if all the orbits are dense.

In the case where G is trivial, assuming Γ is torsion-free, the quotient $X = \Gamma \backslash PSL(2,\mathbb{R})$ becomes the unit tangent bundle $T^1 S$ to the compact hyperbolic surface $S = \Gamma \backslash \mathbb{H}$ obtained from the Poincaré half-plane $\mathbb{H}$. In 1936, G.A. Hedlund [18] proved that the horocycle flow on X is minimal (for an elementary proof, see [14]). In our context, this theorem can be reformulated as follows:

Hedlund's Theorem. *Let Γ be a discrete torsion-free cocompact subgroup of $PSL(2,\mathbb{R})$. Then the right a on $X = \Gamma \backslash PSL(2,\mathbb{R})$ is minimal.*

On the contrary, if X is not compact, M. Kulikov [20] constructed an infinitely generated Fuchsian group without non-empty U-minimal sets. In the case of non uniform lattices of $PSL(2,\mathbb{R})$, like the modular subgroup $PSL(2,\mathbb{Z})$, the U-orbits are dense or periodic. Actually, it is known from [10] that the U-action on X is minimal if and only if X is compact.

When G is not trivial, even assuming X is compact, the U-action may be non minimal. This is the case for example when $G = PSL(2,\mathbb{R})$ and Γ is the product of two cocompact Fuchsian groups. However, in this setting, we prove the following criterion:

Theorem 1. *Let G be a connected Lie group and Γ be a discrete cocompact subgroup of $PSL(2,\mathbb{R}) \times G$. Then the right U-action on $X = \Gamma \backslash PSL(2,\mathbb{R}) \times G$ is minimal if and only if the corresponding $PSL(2,\mathbb{R})$-action is minimal.*

Our proof of Theorem 1 does not use Ratner's famous Orbit-Closure Theorem [25], see also [14] and [19] for an overview. In fact, some ideas will be applied in a non-homogeneous context.

In the second part of this paper, we adopt a foliation point of view, which is natural in the previous context. For any connected Lie group G, the horizontal foliation of $PSL(2,\mathbb{R}) \times G$ by the fibres of the projection on the second factor G is invariant by the action of Γ and so induces a foliation on $X = \Gamma \backslash PSL(2,\mathbb{R}) \times G$ whose leaves are the orbits of the right $PSL(2,\mathbb{R})$-action. In fact, this action gives rise to a *G-Lie foliation* as defined in [16] and [23]. As stated in a theorem by E. Fédida [12], such a foliation is characterised as follows. Given a discrete group Γ acting freely and properly discontinuously on a smooth manifold $\widetilde{M}$, a group homomorphism $h : \Gamma \to G$ and a locally trivial smooth fibration $\rho : \widetilde{M} \to G$ with connected fibres that is Γ-equivariant (i.e. $\rho(\gamma x) = h(\gamma)\rho(x)$ for all