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A functorial formalism for quasi-coherent sheaves on a geometric stack

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Abstract

A geometric stack is a quasi-compact and semi-separated algebraic stack. We prove that the quasi-coherent sheaves on the small flat topology, Cartesian presheaves on the underlying category, and comodules over a Hopf algebroid associated to a presentation of a geometric stack are equivalent categories. As a consequence, we show that the category of quasi-coherent sheaves on a geometric stack is a Grothendieck category.

We also associate, in a 2-functorial way, to a 1-morphism of geometric stacks $f : \mathbf{X} \to \mathbf{Y}$, an adjunction $f^* \dashv f_*$ for the corresponding categories of quasi-coherent sheaves that agrees with the classical one defined for schemes. This construction is described both geometrically in terms of the small flat site and algebraically in terms of comodules over the Hopf algebroid.

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0. Introduction

The theory of quasi-coherent sheaves on algebraic stacks has suffered from an unbalanced record in the published literature. Our initial aim was to develop the basic cohomological properties of quasi-coherent sheaves on algebraic stacks. The corresponding chapters of [29] were not available when we started to study these problems, and the literature presented some gaps. This led us to pursue a general approach to quasi-coherent sheaves with the cohomology formalism in mind. The result is this paper, that has thus a semi-expository nature. Some of the results here are accessible either on unpublished papers or references with different proofs or employing a language that, in our opinion, makes the connection with the classical literature difficult. The paper also contains some new approaches to the functoriality behaviors of quasi-coherent sheaves on algebraic stacks.

The setting of this paper is to define a category of quasi-coherent sheaves through a *small* ringed site by mimicking the original definition given in [12, (5.1.3)] with a view toward an extension of [1] to stacks. As we look toward cohomological properties, we restrict to a certain class of algebraic stacks, the geometric stacks—we will discuss this choice later in the introduction. We will show that to such an algebraic stack $\mathbf{X}$ one can associate an abelian category of $\mathsf{Qco}(\mathbf{X})$ that is Grothendieck, *i.e.* cocomplete with exact filtered direct limits and possessing a generator. Moreover this category agrees trivially with the usual one when $\mathbf{X}$ is equivalent to a scheme. Further we explore the usual variances with respect to 1-morphisms and the corresponding 2-functorial properties.

What are the differences between this approach and the previous ones already in the literature? The very notion of quasi-coherent sheaf on stacks usually does not refer to a ringed site but it is often mistook with a related notion, that of Cartesian *pre*sheaf. Both are equivalent for geometric stacks as we show. The sheaf approach suffers a major technical problem, namely, the lack of functoriality of the *lisse-étale* site.

In [25], a solution is presented for the category of quasi-coherent sheaves defined in [21]. Unfortunately not all the results that we need are contained in this reference. We also develop a treatment in our setting of the adjoint functors $f^* \dashv f_*$ associated to a 1-morphism $f : \mathbf{X} \to \mathbf{Y}$ together with its 2-functorial properties. Note the description in [25, 3.3] of f_* is slightly incorrect because the 1-morphism f would have to be representable by schemes; it is easily repaired by considering the approach in [21] or in [20]. We provide a full detailed construction of both functors. We give also an explicit description of these functors via Hopf algebroids (Corollary 7.7).

We should mention that in the *Stacks Project* [29] a construction similar to ours is made but using *big* flat sites. The drawback of this approach is that in order to the category of sheaves of modules over a big (ringed) site to possess a generator, it is necessary to make the choice of a universe. This construction makes the sites considered automatically functorial but one has to check the invariance of universe. We refrain from using Grothendieck universes by using sets and classes *a la* Von Neumann–Gödel–Bernays but we had to wrestle with the non functoriality of the small sites. This paper thus was developed mostly independently of [29], that is why we refer to [21] for all the basic definitions on stacks.

Outline of the paper. Let us explain first the reason why we assume always that the stacks we consider are quasi-compact and semi-separated, *i.e.* with affine diagonal morphism.