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A pseudo-index approach to fractional equations

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Abstract

The aim of this paper is investigating the existence and multiplicity of weak solutions to non-local equations involving a general integro-differential operator of fractional type, when the nonlinearity is subcritical and asymptotically linear at infinity. More precisely, in presence of an odd symmetric non-linear term, we prove multiplicity results by using a pseudo-index theory related to the genus. As a particular case we derive existence and multiplicity results for non-local equations involving the fractional Laplacian operator. Our theorems, obtained exploiting a novel abstract framework, extend to the non-local setting some results, already known in the literature, in the case of the classical Laplace operator.

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1. Introduction

In the past years there has been a considerable amount of research related to non-resonant elliptic equations. For instance, the semilinear problem

$$\begin{cases} -\Delta u = g(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{D}$$

where Ω is an open bounded domain of $\mathbb{R}^N$ with boundary $\partial\Omega$ and g is a given real function on $\Omega \times \mathbb{R}$ asymptotically linear and possibly odd, has been widely investigated (cf. [1,4,18] and references therein, as well as [8] for the linear case).

Aim of the present work is to provide a multiplicity result for the non-local counterpart of such a problem, whose standard prototype is the fractional Laplacian, that is the equation

$$\begin{cases} -\mathcal{L}_K u = g(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{P}$$

Here Ω is an open bounded domain (with Lipschitz boundary $\partial\Omega$) of $\mathbb{R}^N$ with $N > 2s$, where $s \in]0,1[$ is fixed, and $\mathcal{L}_K$ is the non-local operator defined by

$$\mathcal{L}_K u(x) := \int_{\mathbb{R}^N} \Big(u(x+y) + u(x-y) - 2u(x)\Big) K(y)\, \mathrm{d}y, \quad x \in \mathbb{R}^N,$$

with the kernel $K : \mathbb{R}^N \setminus \{0\} \to]0,+\infty[$ such that:

$$mK \in L^1(\mathbb{R}^N), \quad \textit{where } m(x) = \min\{|x|^2, 1\}; \tag{1.1}$$

$$\textit{there exists } \theta > 0 \textit{ such that } K(x) \geqslant \theta |x|^{-(N+2s)} \textit{ for all } x \in \mathbb{R}^N \setminus \{0\}. \tag{1.2}$$

Let us point out that the Dirichlet datum in (P) is given in $\mathbb{R}^N \setminus \Omega$ (and not just on $\partial\Omega$), according with the non-local nature of $\mathcal{L}_K$. Moreover, we refer to Remark 2.1 for some comments about the role of the above conditions.

In spite of the fact that a lot of papers are concerned with non-local fractional Laplacian equations with superlinear and sublinear growth (cf., e.g., [20,22] and references therein), only very recently this kind of problem has been studied also in non-local setting for asymptotically linear right-hand side (cf. [13]) and here we would like to go further in this direction.

The interest towards equations involving non-local operators has grown more and more, thanks to their intriguing analytical structure and in view of several applications. Indeed, fractional and non-local operators appear in concrete applications in many fields through a new and fascinating scientific approach; see, for instance, the papers [9–11] as a general reference on this topic.

Finding conditions on the data ensuring that problem (P) possesses multiple (weak) solutions is a problem of interest in the current literature. For instance, in this direction, by using an abstract result proved in [19, Theorem 3], the existence of at least three non-trivial solutions has been proved in [16]; see also [15,17] for related arguments and results.

We also observe that very recently the existence and multiplicity of solutions for elliptic equations in the whole space $\mathbb{R}^N$, driven by a non-local integro-differential operator have