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Expo. Math. 33 (2015) 526–534

EXPOSITIONES MATHEMATICAE

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A theorem of Mislin for cohomology of fusion systems and applications to block algebras of finite groups

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Received 10 September 2014; received in revised form 8 November 2014

Abstract

The aim of this short expository article is to give an algebraic proof for a theorem of Mislin in the case of cohomology of saturated fusion systems defined on p-groups when p is odd. Some applications of this theorem, which includes different proofs of known results regarding block algebras of finite groups, are also given.

MSC 2010: 20C20; 16EXX

Keywords: Fusion system; Cohomology; Variety; Block algebra

1. Introduction

A saturated fusion system $\mathcal{F}$ on a finite p-group P is a category whose objects are the subgroups of P and whose morphisms satisfy certain axioms mimicking the behavior of a finite group G having P as a Sylow subgroup. For the convenience of the reader we give, in Section 2, the definition of saturated fusion systems and some basic facts, following [2]. Let k be an algebraically closed field of characteristic p. We denote by $\mathrm{H}^*(G, k)$ the cohomology algebra of a group G with trivial coefficients. We denote by $\mathrm{H}^*(\mathcal{F})$ the subalgebra

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http://dx.doi.org/10.1016/j.exmath.2014.12.006

of $\mathcal{F}$-stable elements in $\mathrm{H}^*(P,k)$, i.e. the cohomology algebra of the saturated fusion system $\mathcal{F}$, which is the subalgebra of $\mathrm{H}^*(P,k)$ consisting of elements $\zeta \in \mathrm{H}^*(P,k)$ such that

$$\mathrm{res}^P_Q(\zeta) = \mathrm{res}_\varphi(\zeta),$$

for any $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q,P)$ and any subgroup Q of P.

A celebrated theorem of Mislin in [14] regarding the control of fusion in group cohomology (stated for compact Lie groups) has now a new short algebraic proof for p odd thanks to Benson, Grodal and Henke [4]. See [7,15] for other algebraic proofs which use Mackey functors and cohomology of trivial source modules; see also [21] for a different algebraic approach. Also, in [9, Remark 5.8] Linckelmann suggests a topological proof for Mislin's theorem in the case of block algebras of finite groups, more precisely for cohomology of fusion systems associated to blocks. We prove this theorem of Mislin in the general context of cohomology of saturated fusion systems for p odd, by extending the proof of Benson, Grodal and Henke to saturated fusion systems.

The k-algebra $\mathrm{H}^*(\mathcal{F})$ is a graded-commutative and finitely generated, hence we associate the spectrum of maximal ideals, i.e. the algebraic variety which we denote by $V_{\mathcal{F}}$. Let $\mathcal{G}$ be a saturated fusion subsystem of $\mathcal{F}$ defined on the same finite p-group P. We have an inclusion map

$$i : \mathrm{H}^*(\mathcal{F}) \to \mathrm{H}^*(\mathcal{G}),$$

which induces a map on varieties

$$i^* : V_{\mathcal{G}} \to V_{\mathcal{F}}.$$

The main result of this paper is the following theorem which contains Mislin's theorem for saturated fusion systems as a special case, when p is odd.

Theorem 1.1. *Let $\mathcal{G}$ be a saturated fusion subsystem of $\mathcal{F}$ defined on the same finite p-group P and p an odd prime. If for each $\zeta \in \mathrm{H}^*(\mathcal{G})$ we have $\zeta^{p^r} \in \mathrm{Im}(i)$ for some $r \geq 0$, then $\mathcal{G} = \mathcal{F}$. In particular we have $\mathrm{H}^*(\mathcal{F}) = \mathrm{H}^*(\mathcal{G})$ if and only if $\mathcal{G} = \mathcal{F}$.*

The ingredients for the proof of the above theorem were already mentioned by Benson, Grodal and Henke in [4, Remark 3.7]. We follow their suggestion and we fill this gap in the literature. There are two ingredients for proving Theorem 1.1. To explain those ingredients, we first introduce the following terminology. Let $\mathcal{G}$ be a saturated fusion subsystem of $\mathcal{F}$ defined on the same finite p-group P. For shortness, we will say that $\mathcal{G}$ *controls p-fusion in $\mathcal{F}$ on elementary abelian p-subgroups* if $\mathrm{Hom}_{\mathcal{G}}(E_1,E_2) = \mathrm{Hom}_{\mathcal{F}}(E_1,E_2)$ for all $E_1, E_2 \leq P$, where E_1, E_2 run over the set of elementary abelian p-subgroups of P. Now one of the ingredients is a property of saturated fusion systems [4, Theorem B] which says that if p is odd and $\mathcal{G}$ controls p-fusion in $\mathcal{F}$ on elementary abelian p-subgroups, then $\mathcal{G} = \mathcal{F}$. Another ingredient is the following theorem which says that control of p-fusion on elementary abelian subgroups happens if and only if i^* is a bijective map. [1, Theorem 2] and [19, Proposition 10.9] are similar statements for group cohomology.

Theorem 1.2. *Let $\mathcal{G}$ be a saturated fusion subsystem of $\mathcal{F}$ defined on the same finite p-group P. Then i^* is surjective. Moreover we have that i^* is an injective map if and only if $\mathcal{G}$ controls p-fusion in $\mathcal{F}$ on elementary abelian p-subgroups.*