Available online at www.sciencedirect.com

ScienceDirect

Expo. Math. 33 (2015) 135–183

EXPOSITIONES MATHEMATICAE

www.elsevier.com/locate/exmath

Local duality for Banach spaces

Manuel González[a], Antonio Martínez-Abejón[b,*]

[a] Departamento de Matemáticas, Universidad de Cantabria, E-39071, Santander, Spain
[b] Departamento de Matemáticas, Universidad de Oviedo, E-33007, Oviedo, Spain

Received 6 November 2012; received in revised form 24 March 2014

Abstract

A local dual of a Banach space X is a subspace of the dual X^* which can replace the whole dual space when dealing with finite dimensional subspaces. This notion arose as a development of the principle of local reflexivity, and it is very useful when a description of X^* is not available.

We give an exposition of the theory of local duality for Banach spaces, including the main properties, examples and applications, and comparing the notion of local dual with some other weaker properties of the subspaces of the dual of a Banach space.

MSC 2010: primary 46B07; secondary 46B08; 46B10

Keywords: Local duality; Local complementation; Finite dual-representability

1. Introduction

In Banach space theory, it is usual to describe the properties of a given Banach space X in terms of its dual space X^*, but many times a representation of the dual space is not available. This is the case for the space $L_\infty(\mu, X)$ of the essentially bounded, measurable, X-valued functions. However, $L_\infty(\mu, X)^*$ contains a natural copy of $L_1(\mu, X^*)$ which can

* Corresponding author.
E-mail address: ama@uniovi.es (A. Martínez-Abejón).

http://dx.doi.org/10.1016/j.exmath.2014.04.002
0723-0869/

replace the whole dual space for many purposes, like norming the elements of $L_\infty(\mu, X)$, or representing the duality on finite dimensional subspaces of $L_\infty(\mu, X)$. Finding those concrete subspaces of the dual space and describing their properties is the aim of the theory of local duality for Banach spaces.

As far as we know, the investigation in local duality began with the *principle of local reflexivity* (P.L.R. for short) which establishes that, when working with finite dimensional subspaces, it is possible to replace the bidual X^{**} by the original space X. The first version of that principle was obtained by Johnson, Rosenthal and Zippin [43], and the current form is due to Lindenstrauss and Rosenthal [51]. The P.L.R. has found many applications in Banach space theory. For example, it implies the existence of a basis for all separable $\mathcal{L}_p$-spaces [51] and for spaces whose dual has a basis [43]. It has also been applied in approximation theory [58], in the study of local complementation of tensor products [10] and in the study of the L-structure of $L_1(\mu)$ [38, A.6]. For an illuminating account of applications of the P.L.R., we suggest [18, Chapter 8], especially its final comments.

Proofs of this principle are given in [4,15,54,67]. Further operator versions can be found in [7] (see also [6]), [8,56,61]. Moreover, the P.L.R. has been also translated to different contexts like that of Banach lattices [9], modules [14], duality of cones [24], operator ideals [60], spaces of compact operators [49] and other non-commutative settings [22,42,62,65].

A closed subspace Z of X^* is a *local dual* of a Banach space X if for every $\varepsilon > 0$ and every pair of finite dimensional subspaces E of X^* and F of X, there exists an operator $L: E \longrightarrow Z$ satisfying the following conditions:

(A) $\big|\|L(x^*)\| - \|x^*\|\big| \leq \varepsilon\|x^*\|$ for all $x^* \in E$,
(B) $L(x^*)\,|_F = x^*\,|_F$ for all $x^* \in E$,
(C) $L(x^*) = x^*$ for all $x \in E \cap Z$.

The principle of local reflexivity exactly says that X, as a subspace of X^{**}, is a local dual of X^*. A similar result, independently obtained in [46,69], is the *principle of local reflexivity for ultrapowers* which says that for every ultrafilter $\mathfrak{U}$, the ultrapower $(X^*)_{\mathfrak{U}}$ is a local dual of the ultrapower $X_{\mathfrak{U}}$. Note that $X = X^{**}$ if and only if X is reflexive, and $(X^*)_{\mathfrak{U}} = (X_{\mathfrak{U}})^*$ for all $\mathfrak{U}$ if and only if X is superreflexive. Further proofs and variations on the P.L.R. for ultrapowers can be found in [5,40,47]; certain ultrapower operator versions can be found in [55,20].

The notion of local duality was inspired by both principles of local reflexivity, and has been developed in several papers like [16,27,32,34–37,66]. For a brief account of these developments we refer to [29].

The purpose of this paper is to present a detailed exposition of the theory of local dual spaces, including many examples and some applications, and emphasizing the relation between the three conditions (A)–(C) that define the concept. In order to do that, we will make a parallel study of several weaker properties that a subspace Z of X^* may have: Z *norming* for X, Z *locally* 1-*complemented* (or ideal) in X^*, and X^* *finite dual representable* in Z. For most of the results, we include proofs that are simpler and more natural than those given in the original papers.

The paper is organized as follows. Section 2 includes some technical results that are useful to work with ε-isometries, a characterization of the norming subspaces of the dual