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Differentiable mappings on products with different degrees of differentiability in the two factors

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Abstract

We develop differential calculus of $C^{r,s}$-mappings on products of locally convex spaces and prove exponential laws for such mappings. As an application, we consider differential equations in Banach spaces depending on a parameter in a locally convex space. Under suitable assumptions, the associated flows are mappings of class $C^{r,s}$.

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1. Introduction and statement of results

This paper gives a systematic treatment of the calculus of mappings on products with different degrees of differentiability in the two factors, called $C^{r,s}$-mappings. We shall develop their basic properties and some refined tools. We study such mappings in an infinite-dimensional setting, which is analogous to the approach to C^r-maps between locally

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convex spaces known as Keller's C^r_c-theory [25] (see [30,24,31,13] and [23] for streamlined expositions, cf. also [4]. For C^r-maps on suitable non-open domains, see [23] and [41]). Some basic facts will be recalled in Section 2.

We first introduce the notion of a $C^{r,s}$-mapping: Let E_1, E_2 and F be locally convex spaces, $U \subseteq E_1$ and $V \subseteq E_2$ be open subsets and $r, s \in \mathbb{N}_0 \cup \{\infty\}$. We say that a map $f : U \times V \to F$ is $C^{r,s}$ if the iterated directional derivatives

$$(D_{(w_i,0)} \cdots D_{(w_1,0)} D_{(0,v_j)} \cdots D_{(0,v_1)} f)(x, y)$$

exist for all $i, j \in \mathbb{N}_0$ with $i \leq r$ and $j \leq s$, and are continuous functions in $(x, y, w_1, \ldots, w_i, v_1, \ldots, v_j) \in U \times V \times E_1^i \times E_2^j$ (see Definition 3.1 for details). To enable choices like $U = [0, 1]$, and also with a view towards manifolds with boundary, more generally we consider $C^{r,s}$-maps if U and V are locally convex (in the sense that each point has a convex neighbourhood) and have dense interior (see Definition 3.2). These properties are satisfied by all open sets. Variants and special cases of $C^{r,s}$-mappings are encountered in many parts of analysis. For example, [2] considers analogues of $C^{0,r}$-maps on Banach spaces based on continuous Fréchet differentiability; see [12, 1.4] for $C^{0,r}$-maps; [14] for $C^{r,s}$-maps on finite-dimensional domains; and [11, p. 135] for certain $\mathrm{Lip}^{r,s}$-maps in the convenient setting of analysis. Cf. also [32,20] for ultrametric analogues in finite dimensions. Furthermore, a key result concerning $C^{r,s}$-maps was conjectured in [15, p. 10]. However the authors' interest concerning the subject was motivated by recent questions in infinite dimensional Lie theory. At the end of this section, we present an overview, showing where refined tools from $C^{r,s}$-calculus are useful.

The first aim of this paper is to develop necessary tools like a version of the Theorem of Schwarz and various versions of the Chain Rule. After that we turn to an advanced tool, the exponential law for spaces of mappings on products (Theorem 3.28). We endow spaces of C^r-maps with the usual compact-open C^r-topology (as recalled in Definition 2.5) and spaces of $C^{r,s}$-maps with the analogous compact-open $C^{r,s}$-topology (see Definitions 3.21 and 4.3). Recall that a topological space X is called a *k-space* if it is Hausdorff and its topology is the final topology with respect to the inclusion maps $K \to X$ of compact subsets of X (cf. [26] and the work [39], which popularized the use of k-spaces in algebraic topology). For example, all locally compact spaces and all metrizable topological spaces are k-spaces. The main results of Section 3 (Theorems 3.25 and 3.28) subsume:

Theorem A. *Let E_1, E_2 and F be locally convex spaces, $U \subseteq E_1$ and $V \subseteq E_2$ be locally convex subsets with dense interior, and $r, s \in \mathbb{N}_0 \cup \{\infty\}$. Then $\gamma^\vee : U \to C^s(V, F)$, $x \mapsto \gamma(x, \bullet)$ is C^r for each $\gamma \in C^{r,s}(U \times V, F)$, and the map*

$$\Phi : C^{r,s}(U \times V, F) \to C^r(U, C^s(V, F)), \quad \gamma \mapsto \gamma^\vee \tag{1.1}$$

is linear and a topological embedding. If $U \times V \times E_1 \times E_2$ is a k-space or V is locally compact, then Φ is an isomorphism of topological vector spaces.

This is a generalization of the classical exponential law for smooth maps. Since C^∞-maps and $C^{\infty,\infty}$-maps on products coincide (see Lemma 3.15, Remark 3.16 and Lemma 3.22), we obtain as a special case that

$$\Phi : C^\infty(U \times V, F) \to C^\infty(U, C^\infty(V, F)) \tag{1.2}$$