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Bounds for Turánians of modified Bessel functions[☆]

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Abstract

Motivated by some applications in applied mathematics, biology, chemistry, physics and engineering sciences, new tight Turán type inequalities for modified Bessel functions of the first and second kinds are deduced. These inequalities provide sharp lower and upper bounds for the Turánian of modified Bessel functions of the first and second kinds, and in most cases the relative errors of the bounds tend to zero as the argument tends to infinity. The chief tools in our proofs are some ideas of Gronwall (1932) on ordinary differential equations, an integral representation of Ismail (1977, 1990) for the quotient of modified Bessel functions of the second kind and some results of Hartman and Watson (1961, 1974, 1983). As applications of the main results some sharp Turán type inequalities are presented for the product of modified Bessel functions of the first and second kinds and it is shown that this product is strictly geometrically concave.

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1. Introduction and preliminaries

1.1. Introduction

Let us denote by I_ν and K_ν the modified Bessel functions of the first and second kinds of real order ν, which are the linearly independent particular solutions of the second order modified Bessel differential equation. For definitions, recurrence formulas and many important properties of modified Bessel functions of the first and second kinds we refer to the classical book of Watson [60]. Recall that the modified Bessel function I_ν, called also sometimes as the Bessel function of the first kind with imaginary argument, has the series representation [60, p. 77]

$$I_\nu(x) = \sum_{n\geq 0} \frac{(x/2)^{2n+\nu}}{n!\Gamma(n+\nu+1)},$$

where $\nu \neq -1, -2, \ldots$ and $x \in \mathbb{R}$. The modified Bessel function of the second kind K_ν, called also sometimes as the MacDonald or Hankel function, is defined as [60, p. 78]

$$K_\nu(x) = \frac{\pi}{2}\frac{I_{-\nu}(x) - I_\nu(x)}{\sin \nu\pi},$$

where the right-hand side of this equation is replaced by its limiting value if ν is an integer or zero. We note that in view of the above series representation $I_\nu(x) > 0$ for all $\nu > -1$ and $x > 0$. Similarly, by using the familiar integral representation [60, p. 181]

$$K_\nu(x) = \int_0^\infty e^{-x\cosh t}\cosh(\nu t)dt,$$

which holds for each $x > 0$ and $\nu \in \mathbb{R}$, one can see that $K_\nu(x) > 0$ for all $x > 0$ and $\nu \in \mathbb{R}$. These functions are among the most important functions of the mathematical physics and have been used (for example) in problems of electrical engineering, hydrodynamics, acoustics, biophysics, radio physics, atomic and nuclear physics, information theory. These functions are also an effective tool for problem solving in areas of wave mechanics and elasticity theory. Modified Bessel functions of the first and second kinds are an inexhaustible subject, there are always more useful properties than one knows. Recently, there has been a vivid interest on bounds for ratios of modified Bessel functions and on Turán type inequalities for these functions. For more details we refer the interested reader to the most recent papers in the subject [4–6,11,34–36,54] and to the references therein. It is important to mention here that surprisingly the existing Turán type inequalities for modified Bessel functions of the first and second kinds appear in many problems of applied mathematics, biology, chemistry, physics and engineering sciences, as we can see in Sections 2 and 3. Motivated by the above applications, in this paper our aim is to reconsider the Turán type inequalities for modified Bessel functions of the first and second kinds. By using some ideas of Gronwall [20] on ordinary differential equations, an integral representation of Ismail [29,30] for the quotient of modified Bessel functions of the second kind, results of Hartman and Watson [25,27,61] and some recent results of Segura [54], in the present paper we make a contribution to the subject and we deduce some new tight Turán type inequalities for modified Bessel functions of the first and second kinds. These