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Positive definite hermitian mappings associated with tripotent elements

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Abstract

We give a simple proof of a significant result used by Y. Friedman and B. Russo in 1985, whose proof was originally based on strong holomorphic results. Here we provide a simple proof, directly deduced from the axioms of JB*-triples, of the fact that for each tripotent e in a JB*-triple E, the bilinear mapping $F_1 : E_1(e) \times E_1(e) \to E_2(e)$, $(x, y) \mapsto F_1(x, y) = \{x, y, e\}$, is definite positive (i.e., $F_1(x, x) \geq 0$ in the JB*-algebra $E_2(e)$ and $F_1(x, x) = 0$ if and only if $x = 0$), where $E_1(e)$ and $E_2(e)$ denote the Peirce-1 and -2 subspaces associated with the tripotent e, respectively.

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1. Introduction

In holomorphic theory, the Riemann mapping theorem states that every simply connected open proper subset of the complex plane is biholomorphically equivalent to the open unit disk. The theory of holomorphic functions of several complex variables is substantially different by many reasons, for example, as noted by Poincaré in the early 1900s, the Riemann mapping theorem fails when the complex plane is replaced by a complex Banach space of higher dimension. Though a complete holomorphic

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classification of bounded simply connected domains in arbitrary complex Banach spaces is unattainable, bounded symmetric domains in finite dimensions were studied and classified by E. Cartan [6] using the classification of simple complex Lie algebras, and by M. Koecher [14] and O. Loos [15] with more recent techniques of Jordan algebras and Jordan triple systems. A domain $\mathcal{D}$ in a complex Banach space X is *symmetric* if for each a in $\mathcal{D}$ there is a biholomorphic mapping $\Phi_a : \mathcal{D} \to \mathcal{D}$; with $\Phi_a = \Phi_a^{-1}$, such that a is an isolated fixed point of Φ_a (cf. [19,7]). In a groundbreaking contribution, W. Kaup shows, in [12], the existence of a set of algebraic–geometric-analytic axioms which determine a class of complex Banach spaces, the class of JB*-triples, whose open unit balls are bounded symmetric domains, and every bounded symmetric domain in a complex Banach space is biholomorphically equivalent to the open unit ball of a JB*-triple; in this way, the category of all bounded symmetric domains with base point is equivalent to the category of JB*-triples (see definitions below).

The dual "holomorphic"–"geometric-analytic" nature of JB*-triples allowed different strategies to prove the most significant results in this category of complex Banach spaces. For example, the contractive projection principle asserting that the class of JB*-triples is stable under contractive projections, was independently proved with holomorphic techniques by W. Kaup [13] and L.L. Stacho [18] and with tools of Functional Analysis by Y. Friedman and B. Russo [10]. Nowadays we know holomorphic and functional analytic techniques to prove most of the significative results in JB*-triple theory, however some important structure results remain unproved with techniques of Functional Analysis. An example of the latter is a useful property stated by Y. Friedman and B. Russo in their study of the structure of the predual of a JBW*-triple carried out in [8, Lemma 1.5]. Before going into details, we recall some background. It follows from the algebraic axioms in the definition of JB*-triples that each tripotent e (i.e. $e = \{e, e, e\}$) in a JB*-triple, E, induces a *Peirce decomposition* of E,

$$E = E_2(e) \oplus E_1(e) \oplus E_0(e),$$

where for $i = 0, 1, 2$ $E_i(e)$ is the $\frac{i}{2}$ eigenspace of the mapping $L(e, e)(x) = \{e, e, x\}$. Triple products between elements in Peirce subspaces satisfy the following Peirce multiplication rules: $\left\{E_i(e), E_j(e), E_k(e)\right\}$ is contained in $E_{i-j+k}(e)$ if $i-j+k \in \{0, 1, 2\}$ and is zero otherwise. In addition,

$$\{E_2(e), E_0(e), E\} = \{E_0(e), E_2(e), E\} = 0.$$

It is further known from the axioms that $E_2(e)$ is a JB*-algebra with product $a \circ_e b := \{a, e, b\}$ and involution $a^{*_e} := \{e, a, e\}$ (cf. [15, Section 3], [19, Sections 19, 21] or [7, Section 1.2 and Remark 3.2.2]). Therefore, the mapping

$$F_1 : E_1(e) \times E_1(e) \to E_2(e), \qquad (x, y) \mapsto F_1(x, y) = \{x, y, e\},$$

is well defined, continuous and sesquilinear. In [8, Lemma 1.5], Friedman and Russo state that F_1 also satisfies the following properties:

(a) F_1 is hermitian, i.e., $F_1(x, y)^{*_e} = Q(e)F_1(x, y) = F_1(y, x)$, for every $x, y \in E_1(e)$;
(b) F_1 is positive definite: $F_1(x, x) \geq 0$ in $E_2(e)$ for every $x \in E_1(e)$, and $F_1(x, x) = 0$ implies $x = 0$ in $E_1(e)$.