Available online at www.sciencedirect.com

ScienceDirect

Expo. Math. 33 (2015) 259–267

EXPOSITIONES MATHEMATICAE

www.elsevier.com/locate/exmath

A direct proof that $\ell_\infty^{(3)}$ has generalized roundness zero

Ian Doust[a], Stephen Sánchez[a], Anthony Weston[b,c,d,*]

[a] *School of Mathematics and Statistics, University of New South Wales, Sydney, NSW 2052, Australia*
[b] *Department of Mathematics and Statistics, Canisius College, Buffalo, NY 14208, USA*
[c] *Faculty of Education and Arts, Australian Catholic University, North Sydney, NSW 2060, Australia*
[d] *Department of Decision Sciences, University of South Africa, PO Box 392, UNISA 0003, South Africa*

Received 28 February 2014; received in revised form 23 April 2014

Abstract

Metric spaces of generalized roundness zero have interesting non-embedding properties. For instance, we note that no metric space of generalized roundness zero is isometric to any metric subspace of any L_p-space for which $0 < p \leq 2$. Lennard, Tonge and Weston gave an indirect proof that $\ell_\infty^{(3)}$ has generalized roundness zero by appealing to non-trivial isometric embedding theorems of Bretagnolle, Dacunha-Castelle and Krivine, and Misiewicz. In this paper we give a direct proof that $\ell_\infty^{(3)}$ has generalized roundness zero. This provides insight into the combinatorial geometry of $\ell_\infty^{(3)}$ that causes the generalized roundness inequalities to fail. We complete the paper by noting a characterization of real quasi-normed spaces of generalized roundness zero.

MSC 2010: primary 46B20; secondary 46B04

Keywords: Generalized roundness; Negative type; L_p-spaces

* Corresponding author at: Department of Mathematics and Statistics, Canisius College, Buffalo, NY 14208, USA.

E-mail addresses: i.doust@unsw.edu.au (I. Doust), stephen.sanchez@unsw.edu.au (S. Sánchez), westona@canisius.edu, westoar@unisa.ac.za (A. Weston).

http://dx.doi.org/10.1016/j.exmath.2014.06.001
0723-0869/© 2014 Elsevier GmbH. All rights reserved.

1. Introduction: negative type and generalized roundness

A notion of generalized roundness for metric spaces was introduced by Enflo [5] (see Definition 1.1(c)). Enflo constructed a separable metric space of generalized roundness zero that is not uniformly homeomorphic to any metric subspace of any Hilbert space. This showed that Hilbert spaces are not universal uniform embedding spaces and thereby settled a question of Smirnov. Enflo's application of generalized roundness to the uniform theory of Banach spaces remains unique and may, indeed, be regarded as an anomaly. The reason for this is that generalized roundness is, for all intents and purposes, an isometric rather than uniform invariant. Indeed, Lennard, Tonge and Weston [8] have shown that the generalized roundness and supremal p-negative type of any given metric space coincide. Negative type is a well-known classical isometric invariant whose origin may be traced back to an 1841 paper of Cayley [3]. We recall the relevant definitions here.

Definition 1.1. Let $p \geq 0$ and let (X, d) be a metric space. Then:

(a) (X, d) has *p-negative type* if and only if for all integers $n \geq 2$, all finite subsets $\{z_1, \ldots, z_n\} \subseteq X$, and all choices of real numbers $\zeta_1, \ldots, \zeta_n$ with $\zeta_1 + \cdots + \zeta_n = 0$, we have:

$$\sum_{i,j=1}^{n} d(z_i, z_j)^p \zeta_i \zeta_j \leq 0. \tag{1.1}$$

(b) p is a *generalized roundness exponent* of (X, d) if and only if for all integers $n > 1$, and all choices of points $x_1, \ldots, x_n, y_1, \ldots, y_n \in X$, we have:

$$\sum_{i,j=1}^{n} \left\{ d(x_i, x_j)^p + d(y_i, y_j)^p \right\} \leq 2 \sum_{i,j=1}^{n} d(x_i, y_j)^p. \tag{1.2}$$

(c) The *generalized roundness* of (X, d) is defined to be the supremum of the set of all generalized roundness exponents of (X, d).

There is a richly developed theory of negative type metrics that has stemmed from classical papers of Cayley [3], Menger [10–12] and Schoenberg [16–18]. Recently there has been intense interest in negative type metrics due to their usefulness in algorithmic settings. A prime illustration is given by the *Sparsest Cut problem with relaxed demands* in combinatorial optimization [7,14]. There are a number of monographs that provide modern, in-depth, treatments of the theory of negative type metrics, including Berg, Christensen and Ressel [1], Deza and Laurent [4], and Wells and Williams [19].

We note that some natural embedding problems involve spaces such as L_p with $0 < p < 1$. These spaces carry the natural quasi-norm $\| \cdot \|_{L_p}$ together with the corresponding quasi-metric $d(x, y) = \|x - y\|_{L_p}$. The terminology being used here, however, is not universal. By a *quasi-metric* d on a set X we mean a function d that satisfies the usual conditions for a metric on X, save that the triangle inequality is relaxed in the following manner: there is a constant $K \geq 1$ such that for all $x, y, z \in X$,

$$d(x, y) \leq K \cdot \{d(x, z) + d(z, y)\}.$$