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# On weakly $D$-differentiable operators

Erik Christensen

*Institut for Matematiske Fag, University of Copenhagen, Denmark*

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**Abstract**

Let $D$ be a self-adjoint operator on a Hilbert space $H$ and $a$ a bounded operator on $H$. We say that $a$ is weakly $D$-differentiable, if for any pair of vectors $\xi, \eta$ from $H$ the function $\langle e^{itD} a e^{-itD} \xi, \eta \rangle$ is differentiable. We give an elementary example of a bounded operator $a$, such that $a$ is weakly $D$-differentiable, but the function $e^{itD} a e^{-itD}$ is not uniformly differentiable. We show that *weak $D$-differentiability* may be characterized by several other properties, some of which are related to the commutator $(Da - aD)$.


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## 1. Introduction

In mathematical physics and especially in quantum mechanics commutators between operators on Hilbert space have played a fundamental role for about 100 years [6]. People are mostly concerned with an expression like $i[D, a]$, where $D$ is an unbounded self-adjoint operator, and $a$ is an operator representing an observable.

Since the appearance of the papers [2,9] by Kadison and Sakai respectively, the research on bounded and unbounded derivations on C*-algebras, [3], took off, and an impressive amount of research has been published. As far as we know, the article [5], was the first which dealt with unbounded derivations on C*-algebras.

It is desirable to be able to perform commutators like $[D, a]$ inside the mathematical discipline *noncommutative geometry* [1]. The reason is, that in Connes' set up of noncom-

*E-mail address:* echris@math.ku.dk.

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mutative geometry the space is replaced by an algebra of operators on a Hilbert space, and derivatives of functions are replaced by commutators of the type $[D, a]$.

Usually the commutator $i[D, a]$ appears as the derivative with respect to the norm topology at $t = 0$ of the function $e^{itD}ae^{-itD}$, and here we meet a part of the problem which inspired this investigation. Actually the norm derivative, or uniform derivative as we prefer to say, is a bounded and everywhere defined operator, whereas the commutator $i(Da - aD)$ is only defined on $\text{dom}(D) \cap \{\xi \in H \,\big|\, a\xi \in \text{dom}(D)\}$, which may not even be a dense subset of $H$.

In order to overcome some of the problems with respect to the domain of definition for sums of unbounded operators, Barry Simon suggested in [10] that one should look at forms rather than at operators. For any bounded operator $a$ on $H$ it is possible to define a sesquilinear form on $\text{dom}(D)$ by

$$\forall \xi, \eta \in \text{dom}(D) : S(i[D, a])(\xi, \eta) := i(\langle a\xi, D\eta\rangle - \langle aD\xi, \eta\rangle).$$

Sometimes this form is bounded and is implemented by a bounded operator $b$ such that

$$\forall \xi, \eta \in \text{dom}(D) : S(i[D, a])(\xi, \eta) = \langle b\xi, \eta\rangle.$$

We have not been able to find a systematic presentation of the results which relate to this phenomenon. We know that many people know many details and aspects of this set up, but we have given up on the project to find out who presented this or that property first. It is our hope that this article may help to clarify and distinguish between some properties, which are related to the function $e^{itD}ae^{-itD}$. We concentrate on various kinds of differentiability at the point $t = 0$, which we list below.

**Definition 1.1.** We say that a bounded operator $a$ is uniformly $D$-differentiable if there exists a bounded operator $b$ on $H$ such that

$$\lim_{t\to 0} \left\| \frac{e^{itD}ae^{-itD} - a}{t} - b \right\| = 0.$$

The operator $b$ is called the uniform $D$-derivative of $a$ and denoted $\delta_u^D(a)$.

We say that a bounded operator $a$ is strongly $D$-differentiable if there exists a bounded operator $b$ on $H$ such that

$$\forall \xi \in H : \lim_{t\to 0} \left\| \Big(\frac{e^{itD}ae^{-itD} - a}{t} - b\Big)\xi \right\| = 0.$$

We say that a bounded operator $a$ is weakly $D$-differentiable if there exists a bounded operator $b$ on $H$ such that

$$\forall \xi, \eta \in H : \lim_{t\to 0} \left| \left\langle \Big(\frac{e^{itD}ae^{-itD} - a}{t} - b\Big)\xi, \eta \right\rangle \right| = 0.$$

The operator $b$ is called the weak $D$-derivative of $a$ and denoted $\delta_w^D(a)$.

The properties above are listed according to strength, with the strongest at the top. The present article is based on the observation that there exists an elementary example, which shows that a weakly $D$-differentiable operator may not be uniformly $D$-differentiable. One