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Round handle decompositions of 1-connected 5-manifolds

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Abstract

An n-dimensional round k-handle is the product of a circle and an ordinary $(n-1)$-dimensional k-handle. This exposition gives an introduction to the topic of round handles, exploring several aspects like the existence of round handle decompositions and their manipulation as well as some applications. As a driving example we show how to explicitly construct minimal round handle decompositions of closed, oriented, simply-connected 5-manifolds, also presenting more specialised results of Shkol'nikov and techniques of Fomenko and Sharko along the way.

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1. Round handles and round handle decompositions

Decomposing manifolds into smaller, better-understood pieces is a useful technique in the study of manifolds. One of the simplest form of decompositions is the structure of a handlebody, which is an analogue of a CW-complex in the category of (smooth) manifolds. Minimal decompositions often provide more immediate information on global topological properties (cf. Remark 1.14). In [2] Asimov introduced the notion of round handles (a generalisation of ordinary handles) to prove the existence of non-singular Morse–Smale flows on certain types of manifolds and showed that closed manifolds with zero Euler characteristic are decomposable into round handles. Shortly after that, Thurston [23] introduced

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round Morse functions while studying codimension 1 foliations and Miyoshi [15] used round surgeries to describe foliated cobordisms. In [22] Stern posed several questions concerning the classification of simply-connected smooth 4-manifolds, some of which were answered by Baykur and Sunukjian [5] using round handles. Round handles also play an essential role in Vogel's construction of Engel structures on parallelisable 4-manifolds [24]. Furthermore, recent adaptations of round surgery suggest further applications in the field of contact geometry (see [1,6]).

The aim of this expository paper is to explore several aspects of round handles—their occurrence, the existence of round handle decompositions and manipulations of those as well as some applications. As a driving example we consider the explicit construction of *minimal* (cf. Definition 1.6) round handle decompositions for closed, oriented, simply-connected 5-manifolds, the search for those being motivated by Barden's classification [4] of these manifolds. In particular, the existence of minimal decompositions for all manifolds in this class will follow. A corresponding statement in terms of round Morse functions was made by Shkol'nikov and can be found in [20]. In general, however, the existence of an exact round Morse function (cf. 2) does not necessarily imply the existence of a minimal round handle decomposition, cf. Example 4.3. Shkol'nikov did not formulate a detailed proof but rather suggested a road map. Our approach adopts the methods proposed in a modified manner to better suit round handles instead of round functions and focuses on the construction of explicit decompositions.

Theorem 1.1. *Any closed, oriented, simply-connected* 5*-manifold admits a minimal round handle decomposition.*

Explicit decompositions of certain building block manifolds (cf. Definition 3.1) are given in Example 4.

In this section we discuss the notion of round handles and their relation to ordinary handles and certain functions by presenting material from [2,23,15] and [7]. In Section 2 we introduce (an adapted version of) techniques of Fomenko and Sharko [7] to explain how round handles can be represented schematically via diagrams. We use these to describe closed, oriented, simply-connected 5-manifolds and show the minimality of certain explicitly constructed round handle decompositions in Section 4. As a preparation, we recapture results of Shkol'nikov [19,20] on Barden's work in Section 3. Finally, Section 5 briefly points out some applications of round handles in topology.

Remark 1.2. Manifolds are understood to be smooth ($\mathcal{C}^\infty$) and connected. They may have non-empty boundary.

Definition 1.3. An *n-dimensional round k-handle* is

$$R_k := S^1 \times D^k \times D^{n-k-1},$$

and its boundary is

$$\partial R_k = S^1 \times S^{k-1} \times D^{n-k-1} \cup S^1 \times D^k \times S^{n-k-2} =: \partial_- R_k \cup \partial_+ R_k.$$

Thus, an n-dimensional round k-handle is the product of a circle and an ordinary $(n-1)$-dimensional k-handle. We call k the *index* of the round handle R_k.