Available online at www.sciencedirect.com

ScienceDirect

Expo. Math. 34 (2016) 95–100

Expositiones Mathematicae

www.elsevier.com/locate/exmath

Hunt's hypothesis (H) and triangle property of the Green function

Wolfhard Hansen[a,*], Ivan Netuka[b]

[a] *Fakultät für Mathematik, Universität Bielefeld, 33501 Bielefeld, Germany*
[b] *Charles University, Faculty of Mathematics and Physics, Mathematical Institute, Sokolovská 83, 186 75 Praha 8, Czech Republic*

Received 10 November 2014

Abstract

Let X be a locally compact abelian group with countable base and let $\mathcal{W}$ be a convex cone of positive numerical functions on X which is invariant under the group action and such that $(X, \mathcal{W})$ is a balayage space or (equivalently, if $1 \in \mathcal{W}$) such that $\mathcal{W}$ is the set of excessive functions of a Hunt process on X, $\mathcal{W}$ separates points, every function in $\mathcal{W}$ is the supremum of its continuous minorants in $\mathcal{W}$, and there exist strictly positive continuous $u, v \in \mathcal{W}$ such that $u/v \to 0$ at infinity.

Assuming that there is a Green function $G > 0$ for X which locally satisfies the triangle inequality $G(x, z) \wedge G(y, z) \leq CG(x, y)$ (true for many Lévy processes), it is shown that Hunt's hypothesis (H) holds, that is, every semipolar set is polar.

MSC 2010: primary 31D05; secondary 60J45; 60J60; 60J75

Keywords: Hunt process; Lévy process; Balayage space; Green function; 3G-property; Continuity principle; Polar set; Semipolar set; Hypothesis (H)

The purpose of this short paper is to show that in the settings considered in [4,5,8,9] Hunt's hypothesis (H) holds, that is, semipolar sets are polar provided the underlying

* Corresponding author.
E-mail addresses: hansen@math.uni-bielefeld.de (W. Hansen), netuka@karlin.mff.cuni.cz (I. Netuka).

http://dx.doi.org/10.1016/j.exmath.2014.12.009

space X is an abelian group and the set $\mathcal{W}$ of positive hyperharmonic functions on X (the set of excessive functions of a corresponding Hunt process) is invariant under the group action. The essential property we use is a local triangle property of a Green function for $(X, \mathcal{W})$. Our results constitute a contribution to the long-lasting discussion of Getoor's conjecture, that is, of the validity of (H) for all "reasonable" Lévy processes (see [3,12] and Example 3).

Let X be a locally compact space with countable base. Let $\mathcal{C}(X)$ denote the set of all continuous real functions on X and let $\mathcal{B}(X)$ be the set of all Borel measurable numerical functions on X. The set of all (positive) Radon measures on X will be denoted by $\mathcal{M}(X)$.

Moreover, let $\mathcal{W}$ be a convex cone of positive lower semicontinuous numerical functions on X such that $(X, \mathcal{W})$ is a balayage space (see [2,6] or [8, Appendix]). In particular, the following holds:

(C) $\mathcal{W}$ linearly separates the points of X, for every $w \in \mathcal{W}$,

$$w = \sup\{v \in \mathcal{W} \cap \mathcal{C}(X) \colon v \le w\},$$

and there are strictly positive $u, v \in \mathcal{W} \cap \mathcal{C}(X)$ such that $u/v \to 0$ at infinity.

Remarks 1. 1. If $1 \in \mathcal{W}$, then there exists a Hunt process $\mathfrak{X}$ on X such that $\mathcal{W}$ is the set $E_{\mathbb{P}}$ of excessive functions for the transition semigroup $\mathbb{P} = (P_t)_{t>0}$ of $\mathfrak{X}$ (see [6, Proposition 1.2.1] and [2, IV.8.1]), that is,

$$\mathcal{W} = \{v \in \mathcal{B}^+(X) \colon \sup_{t>0} P_t v = v\}.$$

2. Let us note that the condition $1 \in \mathcal{W}$ is not very restrictive. Indeed, if $(X, \mathcal{W})$ is a balayage space, $w_0 \in \mathcal{W} \cap C(X)$ is strictly positive, and $\widetilde{\mathcal{W}} := \{w/w_0 \colon w \in \mathcal{W}\}$, then $(X, \widetilde{\mathcal{W}})$ is a balayage space such that $1 \in \widetilde{\mathcal{W}}$, and results for $(X, \widetilde{\mathcal{W}})$ yield results for $(X, \mathcal{W})$.

3. Moreover, given any sub-Markov right-continuous semigroup $\mathbb{P} = (P_t)_{t>0}$ on X such that (C) is satisfied by its convex cone $E_{\mathbb{P}}$ of excessive functions, $(X, E_{\mathbb{P}})$ is a balayage space, and $\mathbb{P}$ is the transition semigroup of a Hunt process (see [6, Corollary 2.3.8] or [8, Corollary A.5]).

Let us recall that, for all $A \subset X$ and $u \in \mathcal{W}$, the function R_u^A is the infimum of all $v \in \mathcal{W}$ such that $v \ge u$ on A, and $\hat{R}_u^A(x) := \liminf_{y \to x} R_u^A(y)$, $x \in X$.

A set P in X is *polar*, if $\hat{R}_v^P = 0$ for some (every) function $v > 0$ in $\mathcal{W}$. A set T in X is *totally thin*, if $\hat{R}_v^T < v$ for some $v \in \mathcal{W}$, and *semipolar*, if it is a countable union of totally thin sets. For example, the sets $\{\hat{R}_u^A < R_u^A\}$, $A \subset X$, $u \in \mathcal{W}$, are semipolar (and subsets of $A \cap \partial A$; see [2, VI.5.11 and VI.2.3]).

A function $h \in \mathcal{B}^+(X)$ is *harmonic* on an open set U in X if $h|_U \in \mathcal{C}(U)$ and $\int h \, d\varepsilon_x^{V^c} = h(x)$, for all $x \in U$ and open V such that $x \in V$ and $\overline{V}$ is compact in U (the measures $\varepsilon_x^{V^c}$ are given by $\int u \, d\varepsilon_x^{V^c} = R_u^{V^c}(x)$, $u \in \mathcal{W}$).

Assumption A. Let us assume that $G \colon X \times X \to (0, \infty]$ is a Borel measurable function, $G = \infty$ on the diagonal, such that G is a Green function for $(X, \mathcal{W})$, that is, the following holds (see [2,6] for the definition of potentials for $(X, \mathcal{W})$):