Available online at www.sciencedirect.com

ScienceDirect

Expo. Math. 33 (2015) 67–77

EXPOSITIONES MATHEMATICAE

www.elsevier.com/locate/exmath

Division point measures from primitive substitutions

Daniel Gonçalves*, Charles Starling

Departamento de Matemática, Universidade Federal de Santa Catarina, 88040-900 Florianópolis SC, Brazil

Received 14 September 2013; received in revised form 28 October 2013

Abstract

In this note we extend results of Olli concerning limits of point measures arising from substitutions. We consider a general primitive substitution on a finite polygon set in $\mathbb{R}^2$ and show that limits of certain atomic measures each converge to Lebesgue measure.

MSC 2010: 28; 37; 51

Keywords: Measure; Division; Iterated primitive substitutions; Tilings

1. Introduction

Associating point measures to iterated substitutions goes back to Kakutani [7] who considered division of the unit interval into two subintervals $[0, \alpha)$ and $[\alpha, 1]$. If one repeats this division (with the same ratio for each interval created) and places a point mass at each division point, the limit of these measures converges weak-$*$ to a measure which is mutually singular with Lebesgue measure unless $\alpha = 1/2$, in which case it equals Lebesgue measure.

In [10], Olli generalized this to a family of substitutions in $\mathbb{R}^2$, namely Conway's pinwheel substitution and Sadun's generalization of it (see [13]). Conway's original pinwheel substitution is a scheme for dividing a $(1, 2, \sqrt{5})$-triangle into five uniformly

* Corresponding author.
E-mail addresses: daemig@gmail.com (D. Gonçalves), slearch@gmail.com (C. Starling).

0723-0869/$ - see front matter
http://dx.doi.org/10.1016/j.exmath.2013.12.009

scaled copies of the original, and Sadun's generalization provides a way to substitute any right triangle into five triangles which are similar to the original, but not uniformly scaled. Since the triangles created by these divisions are similar to the original, the substitution can be repeated, thereby dividing the original triangle into smaller and smaller triangles. At each step k in this process, Olli defines three measures ξ_k, ρ_k and ω_k on the original triangle $\mathcal{R}$ based on the substitution. These generate three sequences of measures that are related to the distribution of vertices and triangles. The measure ξ_k places a point mass at the barycenter of each triangle created at step k, while ρ_k and ω_k place a point mass at the vertices of the triangles. The measure ω_k simply puts the same weight at each vertex, while ρ_k weights each vertex for each triangle which has a vertex which intersects it. Each measure is then normalized so as to be a probability measure. Olli then proves [10, Theorem 1], that each of these measures converges to the same measure in the weak-$*$ topology, and that this limiting measure is equal to (normalized) Lebesgue measure if and only if the original triangle was a $(1, 2, \sqrt{5})$-triangle, that is, when the similar triangles at each step were uniformly scaled copies of $\mathcal{R}$.

We extend this result of Olli to other substitutions with uniform scaling. That is, we consider a finite set $\mathcal{P}$ of polygons in $\mathbb{R}^2$ and a substitution rule ω on $\mathcal{P}$ which divides elements of $\mathcal{P}$ into copies of elements of $\mathcal{P}$ scaled down by a constant factor of $\lambda > 0$. We assume that ω is primitive in the sense that there is a k such that for any $p, q \in \mathcal{P}$ if we divide p, k times, then it will contain a copy of q. This situation is quite common in the theory of substitution tilings such as the Penrose tiling, see [1,14] for example. We assume that the elements of $\mathcal{P}$ are disjoint, and let X be the union of its elements, so that X is a disjoint union of a finite number of polygons embedded in $\mathbb{R}^2$. We assume that the elements of $\mathcal{P}$ are uniformly scaled so that the Lebesgue measure of X is 1. The substitution ω can then be repeatedly applied to the space X, dividing each polygon into smaller ones, and we can define probability measures analogous to those defined in [10]. Again, ξ_k will place a point mass at a consistently chosen internal point of each polygon created at stage k, while ρ_k and σ_k will place point masses at the vertices of the polygons. The measure σ_k puts a single point mass at each vertex, while ρ_k again places weights on the vertices based on how many polygons intersect it (see Definitions 3.1, 3.2, and 3.3 for the precise statements, and see Example 3.4 for a clear picture of the weights that these measures give). We note that we change the name of Olli's ω_k to σ_k because the letter ω is usually reserved for the substitution rule in the literature concerning substitution tilings. We prove the following.

Theorem 1.1. *Each of the sequences $\{\xi_k\}, \{\rho_k\}$ and $\{\sigma_k\}$ converges weak-$*$ to Lebesgue measure.*

This is consistent with [10], as the only case there with a constant scaling factor is in the case of the original pinwheel tiling, and there the measures converge to Lebesgue measure.

This note is organized as follows. In Section 2 we present the background material needed to define our measures, as well as recall the Perron–Frobenius theorem. Since primitive substitutions in $\mathbb{R}^2$ arise most often in the study of aperiodic tilings, we use terminology common to that setting (for instance, we will call the polygons created at each stage *tiles*). In Section 3 we define our measures, and prove that $\{\xi_k\}$ and $\{\rho_k\}$ converge to