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Liouville's theorem and heat kernels

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Abstract

We give a proof of Liouville's theorem for harmonic functions by the method of heat kernels. This method also works for the extension of Liouville's theorem in which Laplace's equation is replaced by a higher-order elliptic equation with constant coefficients.

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Liouville's theorem for harmonic functions is based on the mean-value property (see [1,4,5]). Although the mean-value property has a merit in that the theorem is derived via Harnack's inequality under a weaker assumption that the function is bounded below [3], an additional step of regularization is required for the proof if we consider the function to be a weak solution of Laplace's equation. The purpose of this paper is to present a proof of Liouville's theorem by the method of heat kernels. From a physical viewpoint, our method reflects the fact that there is no more heat exchange after the system of heat diffusion reaches thermal equilibrium.

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Theorem 1 (*Liouville's Theorem*). *Let $u(x)$ be a bounded continuous function on the whole space $\mathbb{R}^d$ and satisfy $\Delta u(x) = 0$ in the distributional sense:*

$$\int_{\mathbb{R}^d} u(x)\Delta\phi(x)\,dx = 0 \quad \text{for all } \phi \in C_0^\infty(\mathbb{R}^d), \tag{1}$$

where $\Delta = \sum_{j=1}^d \partial^2/\partial x_j^2$. Then $u(x)$ is a constant.

Proof. We set

$$K(x) = \frac{1}{(4\pi)^{d/2}} \exp\left(-\frac{|x|^2}{4}\right), \qquad K_t(x) = t^{-d/2} K\left(\frac{x}{\sqrt{t}}\right)$$

for $t > 0$. Then $K_t(x)$ is a function in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ and satisfies the heat equation $(\partial/\partial t)K_t(x) - \Delta K_t(x) = 0$ and $\int_{\mathbb{R}^d} K_t(x)\,dx = 1$. We define the function $v(t,x)$ by

$$v(t,x) = K_t * u(x) = \int_{\mathbb{R}^d} K_t(x-y)u(y)\,dy,$$

which is in $C^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$. We note that (1) also holds for all $\phi \in \mathcal{S}(\mathbb{R}^d)$ by the boundedness of u, since $C_0^\infty(\mathbb{R}^d)$ is dense in $\mathcal{S}(\mathbb{R}^d)$. By the properties of K_t and the assumption on u we have

$$\begin{aligned}\partial_t v(t,x) &= \int_{\mathbb{R}^d} \partial_t\{K_t(x-y)\}u(y)\,dy = \int_{\mathbb{R}^d} \Delta_x\{K_t(x-y)\}u(y)\,dy \\ &= \int_{\mathbb{R}^d} \Delta_y\{K_t(x-y)\}u(y)\,dy = 0.\end{aligned}$$

Hence $v(t,x)$ is independent of t. Since $v(t,x)$ converges to $u(x)$ as $t \to 0^+$, we have

$$v(t,x) = u(x).$$

This implies that u is a C^∞ function. Differentiating in x_j gives

$$\partial_{x_j} u(x) = \int_{\mathbb{R}^d} t^{-1/2}(K_j)_t(x-y)u(y)\,dy$$

with $K_j(x) = \partial_{x_j}K(x)$ and $(K_j)_t(x) = t^{-d/2}K_j(x/\sqrt{t})$, and hence

$$|\partial_{x_j} u(x)| \le t^{-1/2}\|K_j\|_{L^1(\mathbb{R}^d)}\|u\|_{L^\infty(\mathbb{R}^d)}.$$

Letting $t \to \infty$, we obtain $\partial_{x_j} u(x) = 0$ for all $x \in \mathbb{R}^d$ and $1 \le j \le d$. Therefore $u(x)$ is a constant. □

One of the extensions of Liouville's theorem is concerned with a solution of polynomial growth to a higher-order elliptic equation with constant coefficients, and formulated as Theorem 2, whose proof by using the Fourier transformation on tempered distributions and the properties of distributions with compact support is suggested in [2]. The method of heat kernels also works for this extended theorem.