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A useful application of Brun's irrationality criterion

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Abstract

We show that Apéry's sequence of rational numbers that converge to $\zeta(3)$ contains a subsequence that satisfies an irrationality criterion of Brun.

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1. Introduction

In 1910, Viggo Brun established a sufficient condition for the limit of a convergent sequence of rational numbers to be irrational. In a later paper he described the result as "simple but unfortunately not very useful" since very few sequences satisfy the rather stringent criteria. His result is as follows.

Theorem 1 (*Brun's Irrationality Criterion, [1]*)**.** *Let (x_n) and (y_n) be strictly increasing sequences of natural numbers such that (x_n/y_n) is a strictly increasing sequence and tends to some limit L. If the sequence (δ_n) given by*

$$\delta_n = \frac{x_{n+1} - x_n}{y_{n+1} - y_n}$$

is strictly decreasing then L is irrational.

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Whilst Brun did not consider the criterion very useful, it was pointed out in a recent note by Angelo Mingarelli [3] that it may be useful enough to give a new proof of Apéry's theorem that $\zeta(3)$ is irrational. In this paper we confirm Mingarelli's suspicions, proving that the sequences that Apéry defined in order to apply Dirichlet's criterion contain a subsequence that satisfies Brun's criterion. The result requires some of the same steps as Apéry's original proof, in particular the confirmation that his two rational sequences satisfy a certain recurrence relation and have explicitly bounded denominators. We omit reproving these details here, as they are proved mostly through exhausting algebraic manipulation rather than any crucial insights. The curious reader can find both proofs in Alfred van der Poorten's excellent paper on the theorem [5].

In the final part of the paper we will prove a second conjecture of Mingarelli that concerns the sequence δ_n defined above when the two sequences under consideration are Apéry's sequences. Mingarelli conjectured that in this case (δ_n) contains arbitrarily long runs of consecutive decreasing terms. This result is not sufficient to apply Brun's criterion, but is an interesting result nonetheless.

2. A proof of Apéry's theorem

In 1978 Roger Apéry defined a pair of sequences whose ratio converged to $\zeta(3)$ quickly enough to apply Dirichlet's criterion, and thus established the irrationality of $\zeta(3)$. The result came somewhat out of the blue, as did the sequences he defined. They involve an auxiliary function defined for integers $0 \leq k \leq n$ by

$$c_{n,k} = \sum_{\ell=1}^{n} \frac{1}{\ell^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m} \binom{n+m}{m}}.$$

With this he then defined

$$a_n = \sum_{k=0}^{n} c_{n,k} \binom{n}{k}^2 \binom{n+k}{k}^2$$

and

$$b_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2.$$

The terms a_n are not in general integral, but if one defines

$$\nu_n = 2(\mathrm{lcm}[1, 2, \ldots, n])^3$$

then $\nu_n a_n$ and $\nu_n b_n$ are a pair of integer sequences with the property that

$$\left| \zeta(3) - \frac{\nu_n a_n}{\nu_n b_n} \right| \leq \frac{c}{(\nu_n b_n)^{1+\delta}}$$

for all n and some explicit constants $c, \delta > 0$. This is Dirichlet's criterion for irrationality, and so $\zeta(3)$ is irrational.

The definitions above make it clear that $(\nu_n a_n)$ and $(\nu_n b_n)$ are increasing sequences of positive integers, and we will prove later that (a_n/b_n) is also an increasing sequence. So