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Wavelet-type frames for an interval

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Abstract

We construct a sequence of rational functions, which will be called *disc wavelets*, parametrized by points in the unit disc $\lambda_{k,n}(\gamma) = \left(1 - \frac{\gamma}{2^n}\right)(\omega_n)^k$, $n \in \mathbb{N}$, $0 \le k < n$, where $\omega_n = e^{2\pi i/n}$ is the primitive nth root of unity, and obtain a full description, in terms of the parameter γ, of the sets $\{\lambda_{k,n}(\gamma)\}$ yielding frames for the space $L^2(-1, 1)$. This is done using a disc version of the Bargmann transform, which maps $L^2(-1, 1)$ to the classical spaces of the unit disc (Bergman, Hardy, Dirichlet) and applying Seip's description of sampling sets in the unit disc. We also describe how to replace the points $\lambda_{k,n}(\gamma)$ by the orbit of a Fuchsian group Γ, observing that Seip's density of the orbit of a point contained in the fundamental region of a Fuchsian group Γ is equal to m_0, where m_0 is the smallest number among the weights of the automorphic forms with a zero in the fundamental region of the group Γ.

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1. Introduction

Classical orthogonal expansions on the real line can be divided into three classes of objects, according to the set where the corresponding L^2 space is defined: the line $\mathbb{R}$, the

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half-line $\mathbb{R}^+$ and the interval $(-1, 1)$. For instance, using orthogonal polynomials, $L^2(\mathbb{R})$ can be expanded in the Hermite, $L^2(\mathbb{R}^+)$ in the Laguerre and $L^2(-1, 1)$ in the Gegenbauer basis.

In the last part of the twentieth century, new systems of functions, known in physics as coherent and affine coherent states, and in signal analysis as Gabor and wavelet frames, have offered an extremely flexible alternative method for describing functions in $L^2(\mathbb{R})$ and in $L^2(\mathbb{R}^+)$ (see [1,5,4]). However, no such method exists for the description of functions in $L^2(-1, 1)$. It is the purpose of the present article to fill in this gap using a very simple idea which combines elementary facts of orthogonal polynomials with a sampling theorem on Bergman spaces.

We should point out that the material to be presented contains no proofs of hard theorems. It is rather a combination of known results and facts in a very particular form, leading to statements we could not find in the literature. It is supposed to be readable by someone with a non-specialized mathematical background.

1.1. Decomposition of $L^2(\mathbb{R}^+)$ using points from the upper half-plane

Denote the complex upper half-plane by $\mathbb{C}^+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. For $\alpha > -1$, the weighted Bergman space $\mathcal{A}_\alpha(\mathbb{C}^+)$ of the upper half-plane consists of the functions F analytic in $\mathbb{C}^+$ for which

$$\|F\|^2_{\mathcal{A}_\alpha(\mathbb{C}^+)} = \int |F(x+iy)|^2\, y^\alpha dxdy < \infty,$$

the Hardy space $\mathcal{H}^2(\mathbb{C}^+)$ consists of the functions F analytic in $\mathbb{C}^+$ for which

$$\|F\|^2_{\mathcal{H}^2(\mathbb{C}^+)} = \sup_{0<y<\infty} \int_{-\infty}^{\infty} |F(x+iy)|^2\, dx < \infty,$$

and the weighted Dirichlet space $\mathcal{D}_\alpha(\mathbb{C}^+)$ consists of the functions F analytic in $\mathbb{C}^+$ for which

$$\|F\|^2_{\mathcal{D}_\alpha(\mathbb{C}^+)} = \int \left|F'(x+iy)\right|^2 y^\alpha dxdy < \infty.$$

There is a Paley–Wiener theorem for the classical weighted spaces of the upper half-plane [3]. It can be understood as a proposition about the *Bergman transform* of order α,

$$\operatorname{Ber}_\alpha f(z) = \int_0^\infty t^{\alpha-1} f(t) e^{izt} dt, \tag{1.1}$$

stating that the map $\operatorname{Ber}_\alpha$ is a unitary operator

$$\operatorname{Ber}_\alpha : L^2(\mathbb{R}^+) \to \begin{cases} \mathcal{A}_{2\alpha-3}(\mathbb{C}^+) & \alpha > 1 \\ \mathcal{H}^2(\mathbb{C}^+) & \alpha = 1 \\ \mathcal{D}_\alpha(\mathbb{C}^+) & 1/2 < \alpha < 1, \end{cases} \tag{1.2}$$

where standard notations are used for the Bergman, the Hardy and the Dirichlet spaces of the upper half-plane.