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Quasicoherent sheaves on toric schemes

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Abstract

Let X be the toric scheme over a ring R associated with a fan Σ. It is shown that there are a group B, a B-graded R-algebra S and a graded ideal $I \subseteq S$ such that there is an essentially surjective, exact functor $\widetilde{\bullet}$ from the category of B-graded S-modules to the category of quasicoherent $\mathscr{O}_X$-modules that vanishes on I-torsion modules and that induces for every B-graded S-module F a surjection Ξ_F from the set of I-saturated graded sub-S-modules of F onto the set of quasicoherent sub-$\mathscr{O}_X$-modules of $\widetilde{F}$. If Σ is simplicial, the above data can be chosen such that $\widetilde{\bullet}$ vanishes precisely on I-torsion modules and that Ξ_F is bijective for every F. In case R is noetherian, a toric version of the Serre–Grothendieck correspondence is proven, relating sheaf cohomology on X with B-graded local cohomology with support in I.

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0. Introduction

Let R be a commutative ring, let S be a positively $\mathbb{Z}$-graded R-algebra that is generated by finitely many elements of degree 1, and let $S_+ = \bigoplus_{n>0} S_n$ be its irrelevant ideal. We consider the projective R-scheme $X = \mathrm{Proj}(S) \to \mathrm{Spec}(R)$. It is a fundamental and classical result that there is an essentially surjective, exact functor $\widetilde{\bullet}$ from the category of $\mathbb{Z}$-graded S-modules to the category of quasicoherent $\mathscr{O}_X$-modules that vanishes precisely

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on S_+-torsion modules, and that induces for every $\mathbb{Z}$-graded S-module F a bijection between the set of S_+-saturated graded sub-S-modules of F and the set of quasicoherent sub-$\mathscr{O}_X$-modules of $\widetilde{F}$ ([14, II.2]). This allows a reasonable translation between projective geometry and $\mathbb{Z}$-graded commutative algebra. Even more, the well-known Serre–Grothendieck correspondence extends this in some sense to yield a translation between sheaf cohomology on X and $\mathbb{Z}$-graded local cohomology with support in S_+, provided R is noetherian ([4, 20.3]). The aim of this article is to present analogous results for toric schemes (which are not necessarily projective), a natural generalisation of toric varieties.

Recall that a *toric variety* (over the field $\mathbb{C}$ of complex numbers) is a normal, irreducible, separated $\mathbb{C}$-scheme of finite type containing an open torus whose multiplication extends to an algebraic action on the whole variety. If V is an $\mathbb{R}$-vector space of finite dimension, N is a $\mathbb{Z}$-structure on V, M is its dual $\mathbb{Z}$-structure on V^*, and Σ is an N-rational fan in V, then these data define a toric variety as follows: If σ is a cone in Σ, then the set $\sigma^\vee \cap M$ of M-rational points of its dual is a submonoid of finite type of M. Its algebra $\mathbb{C}[\sigma^\vee \cap M]$ corresponds to a variety $X_\sigma(\mathbb{C})$, and the facial relations between the cones in Σ allow to glue these varieties and obtain a toric variety $X_\Sigma(\mathbb{C})$. In fact, one can show that *every* toric variety can be constructed in this way. Besides tori, the class of toric varieties contains affine spaces, projective spaces, weighted projective spaces, or Hirzebruch surfaces, but also singular varieties (e.g., the surface in $\mathbb{C}^3$ defined by the equation $x_1x_2 = x_3^2$, or the hypersurface in $\mathbb{C}^4$ defined by the equation $x_1x_2 = x_3x_4$), and non-projective proper varieties (cf. [8, Appendice]); we refer the reader to [9] or [10] for a lot more examples of toric varieties.

The above construction of a toric variety from a fan can be performed over an arbitrary ring R instead of $\mathbb{C}$, yielding what we call a *toric scheme.* This construction is compatible with base change, and in this sense every toric scheme is obtained by base change from a "universal" toric scheme over $\mathbb{Z}$; this is essentially the viewpoint taken by Demazure in [8], where toric varieties make their first appearance in the literature. Even when one is only interested in toric varieties over $\mathbb{C}$, performing base change and hence considering toric schemes instead of toric varieties seems necessary in order to attack fundamental geometric questions (and is, from a scheme-theoretical point of view, anyway a very natural thing to do). For example, in order to investigate whether the Hilbert scheme of a toric variety $X_\Sigma(\mathbb{C})$ exists one has to study quasicoherent sheaves on the base change $X_\Sigma(R)$ for *every* $\mathbb{C}$-algebra R, and similarly for related representability questions. Somewhat less fancy, using base change and hence toric schemes one can reduce the study of toric varieties to the study of toric schemes defined by fans that are not contained in a hyperplane of their ambient space.

Most of the toric literature treats varieties over $\mathbb{C}$ (or algebraically closed fields), and only rarely over arbitrary fields. In arithmetic geometry, toric schemes over more general bases (mostly valuation rings) appear more often (cf. [17], [20], [6], [19], [15]). The author's article [26] contains a systematic study of basic properties of toric schemes over arbitrary bases.

Now, let $X = X_\Sigma(R)$ be the toric scheme over R associated with a rational fan Σ; for simplicity, we suppose that Σ is not contained in a hyperplane of its ambient space.