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Roots of solvable polynomials of prime degree

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Abstract

An explicit formula for the most general root of a solvable polynomial of prime degree is stated and proved. Such a root can be expressed rationally in terms of a single compound radical determined by the roots of a cyclic polynomial whose degree divides $\mu - 1$, where μ is the prime. The study of such formulas was initiated by a formula of Abel for roots of quintic polynomials that are solvable, and was carried forward by Kronecker and a few others, but seems to have lain dormant since 1924. A formula equivalent to the one given here is contained in a paper of Anders Wiman in 1903, but it seems to have been forgotten.

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1. Introduction

Shortly after Niels Henrik Abel's death in 1829, A.J. Crelle published an excerpt [1] from a letter he had received from Abel in 1826, in which Abel had stated, without explanation, a formula that he claimed represented the most general root of a solvable quintic with rational coefficients. That is, the formula described a quantity in an extension of the field of rational numbers, constructed using radicals, that not only was a root of an irreducible polynomial of degree 5 with rational coefficients, but also contained enough parameters – so Abel claimed – that it could represent, when the parameters were

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correctly chosen, a root of *any* given irreducible polynomial of degree 5 with rational coefficients, provided, of course, that the polynomial was one whose roots were expressible by radicals.

Abel's amazing and baffling assertion probably inspired[1] Leopold Kronecker's similarly amazing paper [8] of 1853, in which he generalized Abel's formula to the case of roots of solvable polynomials of any prime degree. Kronecker's formula accomplished in the general case somewhat less than Abel's formula had accomplished in the quintic case, because it depended upon being able to construct the roots of the most general cyclic polynomial of any given degree, whereas Abel *provided* the roots that were needed in the quintic case (see Appendix A).[2] Kronecker, like Abel, did not prove his assertions.

The formulas of Abel and Kronecker in fact are valid only in what might be considered to be the generic case, the one in which $\nu = \mu - 1$ in the notation used below.[3] Heinrich Weber's *Lehrbuch der Algebra* ostensibly gives a formula that is valid generally, but in fact it only covers this generic case (see Section 6). The first published formula that was valid in all cases was that of Anders Wiman in 1903 [12].

After Robert Fricke's revision [6] of Weber's *Lehrbuch* in 1924, the only investigation of explicit formulas for roots of solvable polynomials of prime degree that I am aware of is my own paper [4] in which I reached a formula equivalent to Wiman's, although I did not realize it at the time. I had found Wiman's proof too difficult to follow, and only recently, while revising and simplifying my own proof and reviewing Wiman's, did I realize that the two formulas agree. The exposition in the present paper shortens and clarifies the one given in [4] and corrects an error in that paper (see Appendix B).

It probably needs to be emphasized that Abel's formula is not "a solution of solvable quintics" in the sense that the quadratic formula is a solution of quadratics, because it does not give an algorithm for going from a given solvable quintic to an expression of its roots in terms of radicals. Instead, it is an expression involving radicals and parameters that has the property that, given any solvable quintic, it is possible to choose values for the parameters in such a way that the quantity is a root of the given quintic. The proof that the formula has this property does *not* provide an algorithm for finding the requisite values of the parameters. (See Appendix C.)

The formula developed here does the same for solvable polynomials of any odd prime degree μ. Section 2 presents an algorithm for constructing solvable *extensions* of prime degree μ of a given ground field K—that is, extensions of K of degree μ in which all quantities can be expressed in terms of radicals. Section 3 proves that every irreducible solvable polynomial of degree μ has a root in some extension constructed in this way.

[1] The link between [1] and Kronecker's work is obscured by the fact that Volume 4 of Kronecker's *Werke* directs the reader to the republication of [1] in the 1881 edition of Abel's *Oeuvres*, and this edition does not cite the original 1830 publication. Kronecker mentions [1] explicitly on the first page of [8], and later in the paper gives a specific reference – to the 1839 edition of Abel's *Oeuvres*, not to the original publication which he surely knew – but he says it was *wenig beachtete* (little noticed).

[2] In [8], Kronecker does make some remarks about this secondary problem of constructing roots of cyclic polynomials, as is mentioned in Section 5. The genesis of class field theory lies in these remarks, but that is another story.

[3] In a later paper, Kronecker gave formulas (see (III) and (IV) in [9]) that closely resemble formula (4.1), which shows, in my opinion, that he understood the general case, at least by 1856. In all likelihood Abel would also have understood the general case.