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Generalized Askey functions and their walks through dimensions

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Abstract

We call Φ_d the class of continuous functions $\varphi : [0, \infty) \to [0, \infty)$ such that the radial function

$$\psi(\mathbf{x}) := \varphi(\|\mathbf{x}\|), \quad \mathbf{x} \in \mathbb{R}^d,$$

is positive definite on $\mathbb{R}^d$, for d a positive integer. We then introduce the *generalized Askey class* of functions $\varphi_{n,k,m}(\cdot) : [0, \infty) \to [0, \infty)$ and show for which values of n, k and m such a class belongs to the class Φ_d. We then show walks through dimensions for scale mixtures of members of the class Φ_d with respect to nonnegative bounded measures; in particular, we show that, for a given member of Φ_d, there exist some classes of measures whose associated scale mixture does not preserve the same isotropy index d and allows us to jump into another dimension d' for the class Φ_d. These facts open surprising connections with the celebrated class of multiply monotone functions.

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1. Introduction

Positive definite radial functions have a long history. They enter as an important chapter in all treatments of harmonic analysis and can be traced back to papers by Caratheodory,

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Herglotz, Bernstein and Matthias, culminating in Bochner's theorem from 1932 to 1933. See Berg [2] or V.P. Gurarii [10] for details. The role of positive definite functions has been discussed in several branches of mathematics and statistics including analysis (Schoenberg [16]), random fields theory and statistics (Yaglom [19]) and numerical analysis (e.g. Wendland [17]).

A complex-valued function $f : \mathbb{R}^d \to \mathbb{C}$ is said to be positive definite on $\mathbb{R}^d$ if the inequality $\sum_{k,j=1}^{n} c_k \bar{c}_j f(\mathbf{x}_k - \mathbf{x}_j) \geq 0$ is satisfied for any finite system of complex numbers $c_1, c_2, \ldots, c_n$ and points $\mathbf{x}_1, \ldots, \mathbf{x}_n$ in $\mathbb{R}^d$. We call Φ_d the class of continuous functions $\varphi : [0, \infty) \to [0, \infty)$ such that the radially symmetric function (termed *isotropic* in the statistical literature)

$$C(\mathbf{x}) := \varphi(\|\mathbf{x}\|), \quad \mathbf{x} \in \mathbb{R}^d,$$

is positive definite on $\mathbb{R}^d$, for d a positive integer, called the *isotropy index* in [4]. Here $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^d$.

Any such C with $C(0) = 1$ (hence also, φ) has at least two interpretations: it is both the characteristic function of a radially symmetric probability measure, and it is the correlation function of some weakly stationary and isotropic Gaussian random field. Whenever $\varphi \in \Phi_d$, then $\varphi \in \Phi_{d-1}$, implying the inclusion (strict) relations (see Gneiting [6] and Daley and Porcu [4]):

$$\Phi_1 \supset \Phi_2 \supset \cdots \supset \Phi_\infty.$$

By Schoenberg's theorem (1938) [16], for every positive integer $d \geq 1$, $\varphi \in \Phi_d$ if and only if there exists a uniquely determined nonnegative finite Borel measure G_d on $[0, +\infty)$ such that

$$\varphi(t) = \int_{[0,\infty)} \Omega_d(rt) G_d(\mathrm{d}r),$$

where $\Omega_d(t) = \mathbb{E}(\exp(it\langle e_1, \eta\rangle))$, $t \geq 0$, e_1 is a unit vector in $\mathbb{R}^d$, and η is a random vector uniformly distributed on the unit spherical shell $\mathbb{S}^{d-1} \subset \mathbb{R}^d$. Daley and Porcu [4] call such a measure G_d a *Schoenberg measure* and analyze the relations between Schoenberg measures through projection operators that allow us to map an element of the class Φ_d into another of the class $\Phi_{d'}$, for $d \neq d'$ positive integers. One of these operators is the Montée $\widetilde{I}$, proposed by Matheron [13] and then revisited in [6] as well as in [17]: for a function $f : [0, \infty) \to \mathbb{R}$ such that the possibly improper infinite integral $\int_{[0,\infty)} uf(u)\mathrm{d}u = \lim_{T\to\infty} \int_0^T uf(u)\mathrm{d}u$ is required to exist and to be finite and nonzero, the Montée $\widetilde{I}$ is defined as

$$\widetilde{I}f(t) = \frac{\int_{[t,\infty)} uf(u)\mathrm{d}u}{\int_{[0,\infty)} uf(u)\mathrm{d}u}, \quad t > 0.$$

Arguments in [6] show that how the Montée operator maps an element of the class Φ_d, for $d \geq 3$, into an element of the class Φ_{d-2}. Under mild regularity conditions, one can define the inverse operator, called *Descente*, that allows for a bijection between the classes Φ_d and Φ_{d+2}. For such operators, Wendland [17] adopts the illustrative term *walks through*