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A unified approach to the integrals of Mellin–Barnes–Hecke type

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Abstract

In this paper, we provide a unified approach to a family of integrals of Mellin–Barnes type using distribution theory and Fourier transforms. Interesting features arise in many of the cases which call for the application of pull-backs of distributions via smooth submersive maps defined by Hörmander. We derive by this method the integrals of Hecke and Sonine related to various types of Bessel functions which have found applications in analytic and algebraic number theory.

MSC 2010: 33C05; 33C10; 33C15

Keywords: Gamma function; Fourier transforms; Pull-backs of distributions

1. Introduction

The theory of distributions and Fourier transforms has been successfully applied in the theory of differential operators to obtain precise asymptotic properties of solutions. In this paper, we look at some applications of distribution theory in the study of special functions by providing a unified approach to a certain class of integrals of a type studied by Mellin and Barnes.

Integrals involving products of gamma functions along vertical lines were studied first by Pincherle in 1888 and an extensive theory was developed by Barnes [2] and Mellin [16]. Cahen [5] employed some of these integrals in the study of the Riemann zeta function and other Dirichlet series. In the spirit of Mellin's theory some of Ramanujan's formulas have been generalized by G.H. Hardy [10, p. 98 ff]. The work of Pincherle provided impetus for the subsequent investigations of Mellin [16] and Barnes [2] on the integral representations

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of solutions of generalized hypergeometric series (see [17, Chapter 16] and the comment on p. 225). A detailed commentary on Pincherle's work [19] set against a historical backdrop is available in [15].

Among the integrals studied by Barnes, the integral formula (3.6) is well known which served as the point of departure for Barnes for his development of the theory of the hypergeometric functions in his seminal paper [2]. A more exotic example (3.18)–(3.19) appeared in a later paper by Barnes. The integrals of Mellin–Barnes also play an important role in the theory of the q-analogues of the hypergeometric functions introduced by Heine in 1847. For a discussion of Mellin–Barnes integrals we refer to [26, p. 286 ff] and the thorough investigations in the recent books [1,18]. For the classical evaluation of these and other integrals of this class see [1] particularly pp. 89–91 and pp. 151–154. A complete account of q-hypergeometric series and their Mellin–Barnes integral representations is available in [9, Chapter 4].

In his work on real quadratic number fields (see [11, p. 349]) Hecke employs the transformation formula (4.1) of a Mellin–Barnes integral which was generalized by Rademacher with a view towards applications to number fields of higher degree (see [20, p. 58]). The integral of Hecke also features in his proof of Hamburger's theorem on the Riemann zeta function (see [11, p. 378]) where its relation to the Bessel function of the third kind, the Hankel functions, is established. Hecke's formula is reminiscent of an integral ((4.7) below) considered by Sonine in his researches on the Bessel function. The formulas of Sonine and Hecke combine to yield an integral representation of the Macdonald function $K_p(x)$ that is occasionally employed in analytic number theory.

We provide a transparent and unified approach to these results in the present paper using Fourier integrals with non-linear phase functions that yield distributions (densities) given by the pull-back of the Dirac delta distribution. We present in Sections 3 and 4 proofs of all the formulas stated in the introduction (besides a few others) in the spirit of Fourier analysis. It turns out that all these formulas fall out of the basic equation proved in Section 2:

$$\int_{-\infty}^{\infty} dt \int_{\mathbb{R}^k} \exp(-it\rho(u_1, u_2, \ldots, u_k))\phi(u_1, \ldots, u_k)\, du_1 \cdots du_k$$
$$= 2\pi \langle \rho^*(\delta_0), \phi \rangle. \tag{1.1}$$

Here $\rho(u_1, u_2, \ldots, u_k)$ is a phase function, ϕ is an arbitrary member of the Schwartz class $\mathcal{S}(\mathbb{R}^k)$ of rapidly decreasing functions (see [12, Chapter VII]), δ_0 denotes the standard Dirac delta distribution and $\rho^*(\delta_0)$ denotes the pull-back of Dirac delta distribution by ρ (see [12, p. 136] or [8, p. 103]). It is useful to write (1.1) in the more suggestive notation

$$\int_{-\infty}^{\infty} \exp(-it\rho(u_1, u_2, \ldots, u_k))\, dt = 2\pi\delta(\rho(u_1, \ldots, u_k) = 0). \tag{1.2}$$

A noteworthy special case of (1.2) is the *Fourier–Gelfand formula* [8, p. 193]:

$$\int_{\mathbb{R}} \exp it(u - v)\, dt = 2\pi\delta(u - v). \tag{1.3}$$

The paper is organized as follows. Section 2 contains the proof of the basic formula (1.1) that we repeatedly use. Section 3 contains the proofs of the formulas of Ramanujan and