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Functional inequalities for modified Bessel functions

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ABSTRACT

In this paper, our aim is to show some mean value inequalities for the modified Bessel functions of the first and second kind. Our proofs are based on some bounds for the logarithmic derivatives of these functions, which are in fact equivalent to the corresponding Turán-type inequalities for these functions. As an application of the results concerning the modified Bessel function of the second kind, we prove that the cumulative distribution function of the gamma–gamma distribution is log-concave. At the end of this paper, several open problems are posed, which may be of interest for further research.

1. Introduction

Let us consider the probability density function $\varphi : \mathbb{R} \to (0, \infty)$ and the reliability (or survival) function $\overline{\Phi} : \mathbb{R} \to (0, 1)$ of the standard normal distribution, defined by

$$\varphi(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \quad \text{and} \quad \overline{\Phi}(u) = \frac{1}{\sqrt{2\pi}} \int_u^\infty e^{-t^2/2} dt.$$

The function $r : \mathbb{R} \to (0, \infty)$, defined by

$$r(u) = \frac{\overline{\Phi}(u)}{\varphi(u)} = e^{u^2/2} \int_u^\infty e^{-t^2/2} dt,$$

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is known in the literature as the Mills ratio [31, Section 2.26] of the standard normal distribution, while its reciprocal $1/r$, defined by $1/r(u) = \varphi(u)/\overline{\Phi}(u)$, is the so-called failure (hazard) rate, which arises frequently in economics and engineering sciences. Recently, among other things, Baricz [12, Corollary 2.6], by using the Pinelis version of the monotone form of l'Hospital's rule (see [35,3,4] for further details), proved the following result concerning the Mills ratio of the standard normal distribution.

Theorem A. *If $u_1, u_2 > u_0$, where $u_0 \approx 1.161527889\ldots$ is the unique positive root of the transcendent equation $u(u^2+2)\overline{\Phi}(u) = (u^2+1)\varphi(u)$, then the following chain of inequalities holds:*

$$\frac{2r(u_1)r(u_2)}{r(u_1)+r(u_2)} \leq r\left(\frac{u_1+u_2}{2}\right) \leq \sqrt{r(u_1)r(u_2)} \leq r(\sqrt{u_1u_2}) \leq \frac{r(u_1)+r(u_2)}{2} \leq r\left(\frac{2u_1u_2}{u_1+u_2}\right). \tag{1}$$

Moreover, the first, second, third, and fifth inequalities hold for all u_1, u_2 positive real numbers, while the fourth inequality is reversed if $u_1, u_2 \in (0, u_0)$. In each of the above inequalities, equality holds if and only if $u_1 = u_2$.

We note here that, since the Mills ratio r is continuous, the second and third inequalities in (1) actually mean that under the aforementioned assumptions the Mills ratio is log-convex and geometrically concave on the corresponding interval. More precisely, by definition, a function $f : [a, b] \subseteq \mathbb{R} \to (0, \infty)$ is log-convex if $\ln f$ is convex, i.e. if for all $u_1, u_2 \in [a, b]$ and $\lambda \in [0, 1]$ we have

$$f(\lambda u_1 + (1-\lambda)u_2) \leq [f(u_1)]^{\lambda}\,[f(u_2)]^{1-\lambda}.$$

Similarly, a function $g : [a, b] \subseteq (0, \infty) \to (0, \infty)$ is said to be geometrically (or multiplicatively) convex if g is convex with respect to the geometric mean, i.e. if for all $u_1, u_2 \in [a, b]$ and $\lambda \in [0, 1]$ we have

$$g\left(u_1^{\lambda}u_2^{1-\lambda}\right) \leq [g(u_1)]^{\lambda}\,[g(u_2)]^{1-\lambda}.$$

We note that if f and g are differentiable then f is log-convex if and only if $u \mapsto f'(u)/f(u)$ is increasing on $[a, b]$, while g is geometrically convex if and only if $u \mapsto ug'(u)/g(u)$ is increasing on $[a, b]$. A similar definition and characterization of differentiable log-concave and geometrically concave functions also holds.

Mean value inequalities similar to those presented above also appear in the recent literature explicitly or implicitly for other special functions, such as the Euler gamma function and its logarithmic derivative (see, for example, the paper [2] and the references therein), the Gaussian and Kummer hypergeometric functions, generalized Bessel functions of the first kind, general power series (see the papers [5,8,9], and the references therein), and Bessel and modified Bessel functions of the first kind (see [13,18,32]).

In this paper, motivated by the above results, we are mainly interested in mean value functional inequalities concerning modified Bessel functions of the first and second kind. The detailed content is as follows. In Section 2, we present some preliminary results concerning some tight lower and upper bounds for the logarithmic derivative of the modified Bessel functions of the first and second kind. These results will be applied in what follows to obtain an interesting chain of inequalities for modified Bessel functions of the first and second kind analogous to (1). To achieve our goal, in Section 2 we present some monotonicity properties of some functions which involve the modified Bessel functions of the first and second kind. Section 3 is devoted to the study of the convexity with respect to Hölder (or power) means of modified Bessel functions of the first and second kind. The results stated here complete and extend the results from Section 2. As an application of our results stated in Section 2, in Section 4 we show that the cumulative distribution function of the three-parameter gamma–gamma distribution is log-concave for arbitrary shape parameters. This result may be useful in problems of information theory and communications. Finally, in Section 5 we present some interesting open problems, which may be of interest for further research.