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Feynman operator calculus: The constructive theory

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ABSTRACT

In this paper, we survey progress on the Feynman operator calculus and path integral. We first develop an operator version of the Henstock–Kurzweil integral, construct the operator calculus and extend the Hille–Yosida theory. This shows that our approach is a natural extension of operator theory to the time-ordered setting. As an application, we unify the theory of time-dependent parabolic and hyperbolic evolution equations. Our theory is then reformulated as a sum over paths, providing a completely rigorous foundation for the Feynman path integral. Using our disentanglement approach, we extend the Trotter–Kato theory.

1. Introduction

At the end of his book on path integrals with Hibbs [16], Feynman states: "Nevertheless, many of the results and formulations of path integrals can be reexpressed by another mathematical system, a kind of ordered operator calculus. In this form many of the results of the preceding chapters find an analogous but more general representation …involving noncommuting variables". Feynman is referring to his 1951 paper [15], in which he introduces his time-ordered operator calculus.

Feynman's basic idea for this calculus is to first lay out spacetime as one would a photographic film and imagine that the evolution of a physical system appears as a picture on this film, in which one sees more and more of the future as more and more of the film comes into view. From this point of view, we see that time takes on a special role in that it orders the flow of the spacetime events that appear. Feynman then suggested that we let time take on this role in the manipulation of noncommuting variables in quantum field theory. He went on to show that this approach allowed him to write down and compute highly complicated expressions in a very fast, efficient and effective manner.

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The paper by Feynman was written after Dyson had shown that, using Feynman's time-ordering ideas, he could relate the Feynman and Schwinger–Tomonaga theories of QED. Indeed, it was the work of Dyson [14] that first brought the power of time-ordering to the larger community. (A very nice introduction to the path integral side of this story along with the way that Feynman used path integral ideas to create his computational methods can be found in the recent survey by Cartier and DeWitt-Morette [7].) In response to the importance of time-ordering in relating the Feynman and Schwinger–Tomonaga theories, Segal [47] suggested that the provision of mathematical meaning for time-ordering is one of the major problems in the foundations for QED.

A number of investigators have attempted to solve this problem using formal methods. Miranker and Weiss [38] showed how the ordering process could be carried out (in a restricted manner) using the theory of Banach algebras. Nelson [41] also used Banach algebras to develop a theory of operants as an alternate (formal) approach. Araki [4], motivated by the interesting paper by Fujiwara [17] (see below), used yet another formal approach to the problem. Other workers include Maslov [37], who used the idea of a T-product as an approach to formally order the operators and developed an operational theory. An idea that is closest to Feynman's was developed by Johnson and Lapidus in a series of papers. Their work can be found in their recent book on the subject [31]. (The recent paper by DeFacio et al. [8] should also be consulted.)

A major difficulty with each approach (other than that of [31]) is the problem of disentanglement, the method proposed by Feynman to relate his results to conventional analysis. Johnson and Lapidus develop a general ordering approach via a probability measure on the parameter space. This approach is also constructive and offers a different perspective on possible frameworks for disentanglement in the Feynman program.

Cartier and DeWitt-Morette [7] point out that, during the early years, few researchers in mathematics or physics investigated the path integral. The same is true with respect to the number of researchers investigating the Feynman operator calculus. To our knowledge, the paper by Fujiwara [17] is the only one by a physicist in the early literature. Fujiwara agrees with the ideas and results of Feynman with respect to the operator calculus, but is critical of what he calls notational ambiguities, and introduces a different approach. "What is wanted, and what I have striven after, is a logical well-ordering of the main ideas concerning the operator calculus. The present study is entirely free from ambiguities in Feynman's notation, which might obscure the fundamental concepts of the operator calculus and hamper the rigorous organization of the disentanglement technique". Fujiwara's main idea was that the Feynman program should be implemented using a sheet of unit operators at every point except at time t, where the true operator should be placed. He called the exponential of such an operator an expansional to distinguish it from the normal exponential so that, loosely speaking, disentanglement becomes the process of going from an expansional to an exponential.

1.1. Our purpose

The purpose of this review is to provide a survey of recent progress on the constructive implementation of Feynman's program for the operator calculus [15]. The theory is constructive in that we use a sheet of unit operators at every point except at time t, where the true operator is placed, so that operators acting at different times actually commute. Thus, our approach is the mathematical embodiment of Fujiwara's suggestion. More importantly, the structure developed allows us to lift all of analysis and operator theory to the time-ordered setting. The major reference on this topic is [18]. The work in [18] was primarily written for researchers concerned with the theoretical and/or mathematical foundations for quantum field theory. A major objective was to prove two important conjectures of Dyson for quantum electrodynamics, namely that, in general, we can only expect the perturbation expansion to be asymptotic, and that the ultraviolet divergence is caused by a violation of the Heisenberg uncertainty relation at each point in time (see also [19]).

As suggested in [18], it is our contention that the correct mathematical formulation of the Feynman operator calculus should at least have the following desirable features:

- It should provide a transparent generalization of current analytic methods without sacrificing the physically intuitive and computationally useful ideas of Feynman [15].