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The Steinberg formula for orbit groups

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Abstract

The aim of this note is to give a direct proof of the fact that the Steinberg formula holds for the Vaserstein symbol and the universal weak Mennicke symbol. We also give an alternate proof of the fact that the Steinberg formula holds for the Vaserstein symbol.

MSC 2000: 13C10; 19G12

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1. Introduction

Throughout this note, R stands for a commutative ring with unity, for $n \geqslant 1$, $M_n(R)$ the set of all $n \times n$ matrices over R and $\mathrm{GL}_n(R)$ the group of invertible $n \times n$ matrices over R. A row $v = (a_1, a_2, \ldots, a_n) \in R^n$ is said to be unimodular of length n, if there is a row $w = (b_1, b_2, \ldots, b_n) \in R^n$ such that $v \cdot w^t = 1$, where w^t stands for the transpose of w. The set of all unimodular rows of length n over R will be denoted by $\mathrm{Um}_n(R)$.

Given $\lambda \in R$, for $i \neq j$, let $E_{ij}(\lambda) = I_n + \lambda e_{ij}$, where I_n denotes the identity matrix and $e_{ij} \in M_n(R)$ is the matrix whose only non-zero entry is 1 at the (i, j)-th position. Such $E_{ij}(\lambda)$ are called elementary matrices. The subgroup of $\mathrm{GL}_n(R)$ generated by $E_{ij}(\lambda)$, is called the elementary subgroup of $\mathrm{GL}_n(R)$ and will be denoted by $E_n(R)$.

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There is a natural action of invertible matrices (via matrix multiplication) on the set of unimodular rows, given explicitly by: $v \in \mathrm{Um}_n(R)$ goes to $v\sigma$, where $\sigma \in \mathrm{GL}_n(R)$. Note that if one has $v \cdot w^t = 1$, for some $w \in R^n$, then $(v\sigma)(w(\sigma^{-1})^t)^t = 1$ and so $v\sigma \in \mathrm{Um}_n(R)$. In particular, $E_n(R)$ the elementary subgroup of $\mathrm{GL}_n(R)$, also acts on $\mathrm{Um}_n(R)$. With this action one can define a relation $\sim_E$ on $\mathrm{Um}_n(R)$: if $v, w \in \mathrm{Um}_n(R)$, then $v \sim_E w$ if $v = w\varepsilon$ for some $\varepsilon \in E_n(R)$. $\sim_E$ is an equivalence relation. Let $\mathrm{Um}_n(R)/E_n(R)$ be the set of equivalence classes of $v \in \mathrm{Um}_n(R)$ under the equivalence $\sim_E$ and we call this *the orbit space of unimodular rows under elementary action.*

The orbit space of unimodular rows under elementary action has been studied by topologists and algebraists. In fact, it was the topologists who initiated this study, with the main motivation for it coming from algebraic topology: if $C(X)$ denotes the ring of continuous real valued functions on a topological space X, then every unimodular vector $v \in \mathrm{Um}_n(C(X))$, $n \geqslant 2$, determines a map

$$arg(v) : X \longrightarrow \mathbb{R}^n - \{0\} \longrightarrow S^{n-1}.$$

(The first is by evaluation, and the second is the standard homotopy equivalence. As $n \geqslant 2$, we may ignore base points.) Clearly, vectors in the same elementary orbit define homotopic maps. We thus get an element $[arg(v)]$ of $[X, S^{n-1}]$, where $[X, S^{n-1}]$ is the set of homotopy classes of continuous maps from X to S^{n-1}. Studying properties of the set $[X, S^{n-1}]$ is crucial for attaching topological invariants to the given space X. In fact, Freudenthal (see [4]) defined a group structure on the set of homotopy classes of continuous maps from a separable, compact metric space X of covering dimension less than or equal to n, to S^n. For the definition of covering dimension see [8], Section 2. (Also see [14], Section 1.) The study of the set of homotopy classes of maps from S^n to a compact and locally contractible space X of arbitrary covering dimension was done by Hurewicz (see [5]). It was Borsuk in 1936, who indicated how to define a group structure in a more general setup, of which the results of Freudenthal and Hurewicz were special cases. In particular, Borsuk [3] indicated that for $n > 1$, the set $[X, S^{n-1}]$ is a group if X is a Hausdorff, normal topological space of covering dimension strictly less than $2n - 3$. This group is called the $(n-1)$st cohomotopy group of X and is denoted by $\pi^{n-1}(X)$. Invention of cohomotopy groups was an important contribution of Borsuk to topology. Cohomotopy groups were studied in detail by Spanier [8] as they provided a powerful tool for the classification of maps of a space into a sphere. In the same paper, he showed that the then known classification theorems could be obtained more simply using the group structure on the set of homotopy classes of maps.

Motivated by these results from algebraic topology, algebraists became interested in understanding the orbit set of unimodular rows under elementary action over a commutative ring R with unity. An appropriate notion of dimension in the algebraic context was that of "stable dimension", which we recall below and which will be used throughout this note.

Let (R_n) stand for the following condition introduced by Bass (cf. [1]): suppose that for every $(b_1, b_2, \ldots, b_n, b_{n+1}) \in \mathrm{Um}_{n+1}(R)$ there exist c_i in R (depending on the b_i) such that $(b_i + c_i b_{n+1})_{1 \leqslant i \leqslant n} \in \mathrm{Um}_n(R)$. Let $\mathrm{sr}(R)$ denote the infimum of natural numbers n such that (R_n) holds. If no such n exists, $\mathrm{sr}(R) = \infty$. $\mathrm{sr}(R)$ is called the stable rank of R. By definition, stable dimension of R (denoted by $\mathrm{sdim}(R)$) is one less than the stable rank.

In the topological situation, an interesting particular case occurred for a finite CW-complex X of dimension (i.e. the maximum dimension of cells of X) $d \geqslant 2$. L.N. Vaserstein