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Chern–Simons classes for a superconnection

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Abstract

In this note we define the Chern–Simons classes of a flat superconnection, $D + L$, on a complex $\mathbb{Z}/2\mathbb{Z}$-graded vector bundle E on a manifold such that D preserves the grading and L is an odd endomorphism of E. As an application, we obtain a definition of Chern–Simons classes of a (not necessarily flat) morphism between flat vector bundles on a smooth manifold. An application of Reznikov's theorem shows the triviality of these classes when the manifold is a compact Kähler manifold or a smooth complex quasi-projective variety in degrees > 1.

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1. Introduction

Suppose $(M, \mathscr{C}_M^\infty)$ is a $\mathscr{C}^\infty$-differentiable manifold endowed with the structure sheaf $\mathscr{C}_M^\infty$ of smooth functions. Let E be a complex $\mathscr{C}^\infty$ vector bundle on M of rank r equipped with a connection ∇. The Chern–Weil theory defines the Chern classes as

$$c_i(E, \nabla) \in H^{2i}_{dR}(M, \mathbb{R}) \quad \text{for } i = 0, 1, \ldots, r$$

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in the de Rham cohomology of M. These classes are expressed in terms of the GL_r-invariant polynomials evaluated on the curvature form ∇^2.

Suppose E has a flat connection; i.e., $\nabla^2 = 0$. Then the de Rham Chern classes are zero. It is significant to define Chern–Simons classes for a flat connection. These are classes in the $\mathbb{R}/\mathbb{Z}$-cohomology and were defined by Chern–Cheeger–Simons in [7,8].

In the supersetting a study of the $GL(r, s)$–invariant functions has been carried out by Berezin [5]. Quillen [20,21] has looked into the case of defining the Chern character of a superconnection $D + L$ on a $\mathbb{Z}/2\mathbb{Z}$-graded complex vector bundle E. Here D is a smooth connection preserving the grading on E and L is an odd endomorphism of E. The differential forms defined by Quillen are obtained from the Chern character str e^{D+L} (here str denotes the supertrace, see (2) below) and have $(1/i!)\mathrm{str}((D + L)^{2i})$ as their term in degree $2i$. This term is obtained by substituting the curvature form of the superconnection in the $GL(r, s)$-invariant polynomial, $P_i := \mathrm{str}(A_1 A_2 \ldots A_i)$ where A_j are supermatrices (see also Remark 3.6). Even though $e^{(D+L)^2}$ is a power series, each degree term is given by a polynomial.

To define differential characters which are unique liftings of the Chern forms (see Section 2.3), we need to look at the standard polynomials P_i. Notice that the forms $P_i((D + L)^{2i})$ are polynomial expressions in the curvature form and the first Chern form is a constant multiple of $P_1((D + L)^2)$.

In this paper we use the standard polynomials P_i above to define the Chern–Simons classes.

With notations as in Section 2, we show the following:

Theorem 1.1. *Suppose $\{\nabla_t\}_t$ is a family of superconnections on a complex $\mathbb{Z}/2\mathbb{Z}$-graded vector bundle E, such that ∇_0 preserves the $\mathbb{Z}/2\mathbb{Z}$-grading. Suppose ∇_{t_0} is flat for some t_0. Then there is a uniquely determined Chern–Simons class*

$$CS_k(E, \nabla_{t_0}) \in H^{2k-1}(M, \mathbb{R}/\mathbb{Z})$$

for $k \geqslant 1$.

In particular this applies to the following situation:

Corollary 1.2. *Suppose M is a smooth manifold. Let E be a complex $\mathbb{Z}/2\mathbb{Z}$-graded vector bundle on M equipped with a superconnection $\nabla = D + L$ such that D preserves the $\mathbb{Z}/2\mathbb{Z}$-grading and L is an odd endomorphism of E. Assume that ∇ is a flat superconnection. Then there exist uniquely determined Chern–Simons classes*

$$CS_k(E, \nabla) \in H^{2k-1}(M, \mathbb{R}/\mathbb{Z})$$

for $k > 0$. Furthermore, if M is a compact Kähler manifold or a smooth complex quasi-projective variety and D itself is a flat smooth connection then $CS_k(E, \nabla) = 0$ in $H^{2k-1}(M, \mathbb{R}/\mathbb{Q})$ for $k > 1$.

The last claim is an application of Reznikov's theorem [23] on rationality of Chern–Simons classes on compact Kähler manifold.

A homomorphism $u : E_0 \to E_1$ between vector bundles on a smooth manifold M that induces an isomorphism over a subset $A \subset M$ corresponds to an element in the relative