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ScienceDirect
Expo. Math. 25 (2007) 67 – 76
EXPOSITIONES MATHEMATICAE
www.elsevier.de/exmath

The ranks of the homotopy groups of a space of finite LS category

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Received 15 September 2005; received in revised form 16 February 2006

Abstract

Let X be a simply connected CW complex with finitely many cells in each degree. The first part of the paper is a report on the different conjectures for the behavior of the sequence $\mathrm{rk}\,\pi_i(X)$. In the second part, we give conditions on the Lusternik–Schnirelmann category of X and on the depth of the Lie algebra $\pi_*(\Omega X)\otimes\mathbb{Q}$ that imply the exponential growth in k of the sequence $\sum_{i=1}^{d}\mathrm{rk}\,\pi_{k+i}(X)$, for some d.

MSC 2000: 55P35; 55P62; 17B70

Keywords: Homotopy Lie algebra; Graded Lie algebra; Exponential growth; LS category

1. Introduction

Throughout this paper X will denote a 1-connected non-contractible CW complex X with finitely many cells in each degree. Such a space has homotopy groups of the form

$$\pi_i(X)=\mathbb{Z}^{\rho_i}\oplus T_i,$$

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0723-0869/$ - see front matter
doi:10.1016/j.exmath.2006.04.002

where T_i is finite abelian and $\rho_i = \text{rank}\, \pi_i(X)$ is finite. While in general the $\pi_i(X)$ can be arbitrary, their ranks, $\text{rk}\, \pi_i(X)$, satisfy remarkable regularity conditions when the rational Lusternik–Schnirelmann category, $\text{cat}_0\, X$, is finite.

The LS category of X, $\text{cat}\, X$, is the least m (or ∞) such that X is the union of $m+1$ open sets, each contractible in X; $\text{cat}_0\, X$ is the LS category of the rationalization X_0 and satisfies $\text{cat}_0\, X \leqslant \text{cat}\, X$ [1]. Both $\text{cat}_0\, X$ and $\text{cat}\, X$ depend only on the homotopy type of X.

The spaces of LS category $\leqslant 1$ are the co-H-spaces and a space has rational category $\leqslant 2$ if and only if it has the rational homotopy type of the homotopy cofiber of a map between suspensions [2].

Suppose $\text{cat}_0\, X < \infty$. The dichotomy theorem [3] asserts that either X is (rationally) elliptic, that is $\dim \pi_*(X) \otimes \mathbb{Q} < \infty$, or the sequence $\sum_{i=1}^{k} \text{rk}\, \pi_i(X)$ grows exponentially in k. In this latter case X is called (rationally) hyperbolic.

The analysis of the behavior of the sequence $\text{rk}\, \pi_i(X)$ begins with the observation that $\pi_{i+1}(X) \cong \pi_i(\Omega X)$, ΩX denoting the loop space of X. Moreover [4] $L_X = \pi_*(\Omega X) \otimes \mathbb{Q}$ is naturally a graded Lie algebra whose universal enveloping algebra UL_X is naturally isomorphic with $H_*(\Omega X; \mathbb{Q})$. This makes it possible to use Lie theoretic methods to study the sequence $\text{rk}\, \pi_i(X)$, and two invariants that play an important role are the global dimension and depth of a graded Lie algebra L:

$$\text{gldim}\, L = \max\{k \mid \text{Ext}^k_{UL}(\Bbbk, \Bbbk \neq 0\}$$

and

$$\text{depth}\, L = \min\{k \mid \text{Ext}^k_{UL}(\Bbbk, UL) \neq 0\}.$$

It is trivial to see that $\text{depth}\, L \leqslant \text{gldim}\, L$.

Now if $\text{depth}\, L_X = 0$ then L_X is finite dimensional and concentrated in odd degrees. This implies that the Sullivan model of X is a polynomial algebra with differential zero and so $H^*(X; \mathbb{Q})$ is a polynomial algebra and $\text{cat}_0\, X = \infty$. This is the simple part of the following theorem [5], which is a key result in the analysis of the sequence $\text{rk}\, \pi_i(X)$:

If $\text{cat}_0\, X = m < \infty$, *then*

$$1 \leqslant \text{depth}\, L_X \leqslant \text{cat}_0\, X, \quad \textit{and}$$

If $\text{depth}\, L_X = \text{cat}_0\, X$ *then*

$$\text{depth}\, L_X = \text{gldim}\, L_X.$$

The notion of *inert element* has also played an important role in the study of the sequence $\text{rk}\, \pi_i(X)$. An element x in a graded Lie algebra, L, is *inert* if two conditions both hold:

- the ideal I generated by x is free as a graded Lie algebra, and
- the natural map $U\, L/I \to I/[I, I]$, $\alpha \mapsto \alpha \cdot x$, is an isomorphism.

Inert elements are studied in [6], where they are called strongly free, and in [7]. If x is inert then as a graded vector space L is isomorphic to the free product of L/I with the free