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On the generalized Riemann–Hilbert problem with irregular singularities

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Abstract

We study the generalized Riemann–Hilbert problem, which extends the classical Riemann–Hilbert problem to the case of irregular singularities. The problem is stated in terms of generalized monodromy data which include the monodromy representation, the Stokes matrices and the true Poincaré rank at each singular point. We give sufficient conditions for the existence of a linear differential system with such data. These conditions are in particular fulfilled when the monodromy representation is irreducible, as in the classical case. We solve the problem almost completely in dimension two and three. Our results have applications in differential Galois theory. We give sufficient conditions for a given linear algebraic group G to be the differential Galois group over $\mathbb{C}(z)$ of a differential system with the smallest possible number of singularities, and with singularities all Fuchsian but one, at which the Poincaré rank is minimal.

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1. Introduction

There are many approaches to differential equations. One may focus on the existence and behaviour of the solutions, or on algebraic properties of their symmetries. One may also ask for the existence of differential equations that satisfy specific inverse problems such as the Riemann–Hilbert problem, the Birkhoff standard form problem or the inverse problem in differential Galois theory. This article is an attempt to relate the three problems through the statement and solutions of the generalized Riemann–Hilbert problem.

The classical Riemann–Hilbert problem asks for conditions under which a given representation

$$\chi : \pi_1(\mathbb{P}^1(\mathbb{C})\backslash\mathscr{D}, z_0) \longrightarrow \mathrm{GL}(p, \mathbb{C})$$

of the fundamental group of the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ punctured at each point of a finite subset $\mathscr{D}$ not containing z_0, can be realized as the monodromy representation of a linear differential system with Fuchsian singularities only, all in $\mathscr{D}$. Let us recall that a point $a \in \mathscr{D}$ is a *Fuchsian* singularity of a linear differential system $\mathrm{d}y/\mathrm{d}z = B(z)y$, where B is an $n \times n$ matrix with coefficients in $\mathbb{C}(z)$, if a is a simple pole of B (*modulo* a Möbius transformation if $a = \infty$). This problem is still open, although important results of A. Bolibruch [1–4] have reduced it considerably. Several authors have given sufficient conditions either to solve the problem or to construct counterexamples. A. Bolibruch [1] and V. Kostov [5] have shown independently that the irreducibility of the representation χ is a sufficient condition. In dimension two the problem always has a solution (cf. [6]) and in dimension three and four it has been completely elucidated [6,2,7]. In [8], H. Esnault and E. Viehweg have extended the problem over curves of genus $\geqslant 1$ and solved it for irreducible representations of the fundamental group. The Riemann–Hilbert problem is related to problems in many areas of mathematical physics and has become a trend of research over the last 20 years. There is extensive literature available on the subject, in particular on Painlevé equations and isomonodromic deformations. For recent results in this field we refer to [9–16] and [63].

Closely related to the Riemann–Hilbert problem, the Birkhoff inverse problem asks the following. Consider a differential system $z\,\mathrm{d}y/\mathrm{d}z = A(z)y$, where the matrix $A(z) = z^r \sum_{n=0}^{\infty} A_n z^{-n}$ is meromorphic at infinity. Does there exist a differential system $z\,\mathrm{d}y/\mathrm{d}z = B(z)y$, where $B(z)$ is a polynomial coefficient matrix, meromorphically equivalent to the given system and with a Poincaré rank at infinity not greater than the original one? In dimension two and three, the problem is known to have a positive answer, see [17,18], but in higher dimension, although many sufficient conditions have been given, see [3,19,20], the problem remains open in general. The differential systems in Birkhoff standard form appear in complex algebraic geometry in the study of particular Frobenius manifolds, see [20] and references therein.

In the present paper, we extend both the Riemann–Hilbert problem and the Birkhoff standard form problem to the case of an arbitrary number of irregular singularities. We define generalized monodromy data, consisting of the monodromy representation with respect to prescribed singularities and of further prescribed local data at each singularity. These data include the Poincaré rank and Stokes data. The generalized Riemann–Hilbert problem is the following: *Let singular points and generalized monodromy data be given in which all Poincaré ranks are minimal. Construct a system on* $\mathbb{P}^1(\mathbb{C})$ *with these data*. We give sufficient