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# Analyticity and naturality of the multi-variable functional calculus

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## Abstract

Mackey-complete complex commutative continuous inverse algebras generalize complex commutative Banach algebras. After constructing the Gelfand transform for these algebras, we develop the functional calculus for holomorphic functions on neighbourhoods of the joint spectra of finitely many elements and for holomorphic functions on neighbourhoods of the Gelfand spectrum. To this end, we study the algebra of holomorphic germs in weak*-compact subsets of the dual. We emphasize the simultaneous analyticity of the functional calculus in both the function and its arguments and its naturality. Finally, we treat systems of analytic equations in these algebras.


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## 1. Introduction

A continuous inverse algebra is a locally convex unital associative algebra in which the set of invertible elements is open and inversion is continuous. Such an algebra is called Mackey-complete if every smooth curve has a weak integral. This weak completeness property can also be defined in terms of the bounded structure, or in terms of the convergence of special Cauchy sequences.

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Continuous inverse algebras were introduced by Waelbroeck [48]. They play a role in non-commutative geometry, in particular in $K$-theory [7,10,14,37], and in the theory of pseudo-differential operators [23]. Currently, they are attracting attention in the theory of Lie groups and algebras of infinite dimension [19]. They appear as coordinate algebras in root-graded locally convex Lie algebras [35]. Linear Lie groups are most naturally defined as subgroups of continuous inverse algebras. These algebras have advantages over Banach algebras (which are a special case), for instance because they lead to a rich supply of central extensions of Lie groups and algebras [33,34]. This is due to the fact that unlike semi-simple commutative Banach algebras, commutative continuous inverse algebras can have non-zero derivations. A typical example is the algebra of smooth functions on a compact manifold.

In other ways, continuous inverse algebras are strikingly similar to Banach algebras. In particular, the theory of the Gelfand spectrum and the holomorphic functional calculus, which are probably the most important tools for the study of commutative Banach algebras, can be worked out for complex commutative continuous inverse algebras. This is the purpose of the present paper. Section 2 establishes the basic properties of spectra and treats the Gelfand transform. Section 3 is a brief introduction to the differential calculus on locally convex vector spaces which we use. Section 4 develops the functional calculus for holomorphic functions on neighbourhoods of the joint spectrum of finitely many elements in a complex commutative continuous inverse algebra. Since it is based on Cauchy's integral formula, we have to assume that the algebra is Mackey-complete. Section 5 studies the algebra of holomorphic germs in a compact subset of the weak$^*$-dual of a locally convex complex vector space $E$. The most obvious topology on this algebra, the locally convex direct limit topology, has to be modified in order to make the algebra multiplication continuous. Section 6 shows that the second topology differs from the first unless $E$ has countable dimension. These two sections prepare Section 7, which is devoted to the functional calculus for holomorphic functions on neighbourhoods of the Gelfand spectrum in the weak$^*$-dual of a complex commutative continuous inverse algebra. Section 8 treats systems of analytic equations in complex commutative continuous inverse algebras.

In the special case of commutative Banach algebras, the main results of Sections 2–4 and 7 are known. Here a new aspect is the analyticity of the functional calculus map $(f, a) \mapsto f[a]$. Naturality of the functional calculus with respect to algebra homomorphisms is a consequence which may not have received the attention it deserves. Sections 5 and 6 treat the algebra of holomorphic germs in a weak$^*$-compactum $K$ as the direct limit of the system of Banach algebras of bounded holomorphic functions on open neighbourhoods of $K$. This directed system is uncountable in general, so that we have to get by without the powerful theory of countable direct limits, which one uses in the framework of metrizable vector spaces (cf. [5,22]). This is the main progress in these two sections. In Section 8, the classic result on the solution of analytic equations is simplified by the use of implicitly defined holomorphic functions on the weak$^*$-dual of the algebra and of the corresponding functional calculus. This approach allows us to treat systems of analytic equations with the same ease.

These results require a development of the theory which also applies to continuous inverse algebras. For this purpose, the original approach due to Šilov [43], Arens and Calderón [3], and Waelbroeck [46] seems more suitable than the approach by Bourbaki [13]. When Waelbroeck developed his theory in detail [47], even for complete commutative continuous inverse algebras, the joint spectrum which he used was larger than its modern version.