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Expo. Math. 25 (2007) 181 – 186
EXPOSITIONES MATHEMATICAE
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# On Turán's inequality for Legendre polynomials

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Received 22 June 2006; received in revised form 5 October 2006

## Abstract

Let

$$\Delta_n(x) = P_n(x)^2 - P_{n-1}(x)P_{n+1}(x),$$

where $P_n$ is the Legendre polynomial of degree $n$. A classical result of Turán states that $\Delta_n(x) \geqslant 0$ for $x \in [-1, 1]$ and $n = 1, 2, 3, \ldots$ . Recently, Constantinescu improved this result. He established

$$\frac{h_n}{n(n+1)}(1 - x^2) \leqslant \Delta_n(x) \quad (-1 \leqslant x \leqslant 1; \;\; n = 1, 2, 3, \ldots),$$

where $h_n$ denotes the $n$th harmonic number. We present the following refinement. Let $n \geqslant 1$ be an integer. Then we have for all $x \in [-1, 1]$

$$\alpha_n(1 - x^2) \leqslant \Delta_n(x)$$

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doi:10.1016/j.exmath.2006.11.001

with the best possible factor

$$\alpha_n = \mu_{[n/2]}\mu_{[(n+1)/2]}.$$

Here, $\mu_n = 2^{-2n}\binom{2n}{n}$ is the normalized binomial mid-coefficient.


***MSC 2000:*** 26D07; 33C45

***Keywords:*** Legendre polynomials; Turán's inequality; Normalized binomial mid-coefficient

## 1. Introduction

The Legendre polynomial of degree $n$ can be defined by

$$P_n(x) = \frac{1}{n!2^n}\frac{d^n}{dx^n}(x^2-1)^n \quad (n = 0, 1, 2, \ldots)$$

which leads to the explicit representation

$$P_n(x) = \frac{1}{2^n}\sum_{v=0}^{[n/2]}(-1)^v\frac{(2n-2v)!}{v!(n-v)!(n-2v)!}x^{n-2v}.$$

(As usual, $[x]$ denotes the greatest integer not greater than $x$.) The most important properties of $P_n(x)$ are collected, for example, in [1,16]. Legendre polynomials belong to the class of Jacobi polynomials, which are studied in detail in [3,13]. These functions have various interesting applications. For instance, they play an important role in numerical integration; see [12].

The following beautiful inequality for Legendre polynomials is due to P. Turán [15]:

$$\Delta_n(x) = P_n(x)^2 - P_{n-1}(x)P_{n+1}(x) \geqslant 0 \quad \text{for } -1 \leqslant x \leqslant 1 \text{ and } n \geqslant 1.\text{[1]} \tag{1.1}$$

This inequality has found much attention and several mathematicians provided new proofs, far-reaching generalizations, and refinements of (1.1). We refer to [8,9,11,14] and the references given therein.

In this paper we are concerned with a remarkable result published by E. Constantinescu [7] in 2005. He offered a new refinement and a converse of Turán's inequality. More precisely, he proved that the double-inequality

$$\frac{h_n}{n(n+1)}(1-x^2) \leqslant \Delta_n(x) \leqslant \frac{1}{2}(1-x^2) \tag{1.2}$$

is valid for $x \in [-1, 1]$ and $n \geqslant 1$. Here, $h_n = 1 + 1/2 + \cdots + 1/n$ denotes the $n$th harmonic number.

[1] A nice anecdote about Turán reveals that he used (1.1) as his 'visiting card'; see [4].