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Constacyclic codes over finite local Frobenius non-chain rings with nilpotency index 3

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ABSTRACT

The main results of this paper are in two directions. First, the family of finite local Frobenius non-chain rings of length 4 (hence of nilpotency index 3) is determined. As a by-product all finite local Frobenius non-chain rings with p^4 elements, (p a prime) are given. Second, the number and structure of γ-constacyclic codes over finite local Frobenius non-chain rings with nilpotency index 3, of length relatively prime to the characteristic of the residue field of the ring, are determined.

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1. Introduction

After the work of R. Hammons et al. (see [4]) the study of linear codes over finite rings has been an interesting research topic. Results in several directions including the description of structural properties of codes over several families of rings, particularly finite fields and finite chain rings, are available in the literature. The γ-constacyclic codes where γ is a unit of the finite ring A taken as the alphabet, i.e., those codes invariant under the mapping $\sigma_\gamma : \mathrm{A}^n \longrightarrow \mathrm{A}^n$ given by $\sigma_\gamma(a_0, a_1, \ldots, a_{n-1}) = (\gamma a_{n-1}, a_0, \ldots, a_{n-2})$ are a generalization of cyclic codes. The 1-constacyclic codes are the usual cyclic codes and (-1)-constacyclic codes are the negacyclic codes. Finite Frobenius rings represent an interesting family of rings in Coding theory due to the fact that MacWilliams identities on the weight enumerator polynomial of a linear code are satisfied (see [13]). A finite Frobenius ring can be expressed as a direct sum of finite Frobenius local rings, (see [13]), and this induces a decomposition of the linear codes over Frobenius local rings. Finite chain rings are a subfamily of the family of finite Frobenius local rings and γ-constacyclic codes over finite chain rings has been considered by several researchers (see [2,3,11]), so it would be interesting to study γ-constacyclic codes over finite local Frobenius non-chain rings (γ a unit of the ring).

If p is a prime number, it is well-known that up to isomorphism there is only one local ring with p elements, namely the Galois field: 1) $\mathrm{GF}(p)$. The local rings with p^2 elements are: 2) $\mathrm{GF}(p^2)$, 3) $\mathbb{Z}_{p^2}$ and 4) $\mathrm{GF}(p)[\mathrm{X}]/\langle \mathrm{X}^2 \rangle$. If p is odd, the local rings with p^3 elements are: 5) $\mathrm{GF}(p^3)$, 6) $\mathbb{Z}_{p^3}$, 7) $\mathrm{GF}(p)[\mathrm{X}]/\langle \mathrm{X}^3 \rangle$, 8) $\mathbb{Z}_{p^2}[\mathrm{X}]/\langle \mathrm{X}^2 - p, p\mathrm{X} \rangle$, 9) $\mathbb{Z}_{p^2}[\mathrm{X}]/\langle \mathrm{X}^2 - \zeta p, p\mathrm{X} \rangle$, where $\bar{\zeta}$ is a primitive element of $\mathrm{GF}(p)$, 10) $\mathrm{GF}(p)[\mathrm{X}, \mathrm{Y}]/\langle \mathrm{X}, \mathrm{Y} \rangle^2$ and 11) $\mathbb{Z}_{p^2}[\mathrm{X}]/\langle \mathrm{X}^2, p\mathrm{X} \rangle$. If $p = 2$, the local rings with 2^3 elements are: 12) $\mathrm{GF}(2^3)$, 13) $\mathbb{Z}_{2^3}$, 14) $\mathrm{GF}(2)[\mathrm{X}]/\langle \mathrm{X}^3 \rangle$, 15) $\mathbb{Z}_{2^2}[\mathrm{X}]/\langle \mathrm{X}^2 - 2, 2\mathrm{X} \rangle$, 16) $\mathrm{GF}(2)[\mathrm{X}, \mathrm{Y}]/\langle \mathrm{X}, \mathrm{Y} \rangle^2$ and 17) $\mathbb{Z}_{2^2}[\mathrm{X}]/\langle \mathrm{X}^2, 2\mathrm{X} \rangle$, (see [9]). The rings 10), 11), 16) and 17) are not Frobenius local rings, because the annihilator of their maximal ideal is not a simple ideal, and the other rings are chain rings. In [5] the ring $\mathrm{GF}(2)[\mathrm{X}, \mathrm{Y}]/\langle \mathrm{X}^2, \mathrm{Y}^2 \rangle$ with 2^4 elements, which is a local Frobenius non-chain ring was introduced and linear cyclic codes over this alphabet were considered. Recently, finite local Frobenius non-chain rings with 2^4 elements were determined and linear codes over these rings were studied [7]. Now if $p > 2$ is a prime it would be interesting to determine the family of finite local Frobenius non-chain rings with p^4 elements.

The relation $|\mathrm{M}| = |\mathrm{GF}(p^d)|^{\ell_{\mathrm{A}}(\mathrm{M})}$, (see section 2), where M is an A-module over a finite local ring A with residue field $\mathrm{GF}(p^d)$ and $\ell_{\mathrm{A}}(\mathrm{M})$ is the length of M, and the fact that a local ring of length one and two are chain rings, (see section 2), implies that if A is a finite local Frobenius non-chain ring with p^4 elements then A has length 4. On the other hand the family of finite local Frobenius non-chain rings of nilpotency index 3 is large, including finite local Frobenius non-chain rings of length 4 and the rings $\mathrm{A}_{(l,p^d)} = \mathrm{GF}(p^d)[\mathrm{X}_1, \ldots, \mathrm{X}_l]/\langle \mathrm{X}_i\mathrm{X}_j - \mathrm{X}_1\mathrm{X}_2, \mathrm{X}_1^2, \ldots, \mathrm{X}_l^2 : (i,j), 1 \leq i < j \leq l, (i,j) \neq (1,2) \rangle$, where p is a prime and $l \geq 3$ an integer such that $(l-1, p) = 1$. Since the length of the ring $\mathrm{A}_{(l,p^d)}$ is $l + 2$, this family contains rings of all lengths.