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Proof of a conjecture on monomial graphs ☆

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A B S T R A C T

Let e be a positive integer, p be an odd prime, $q = p^e$, and $\mathbb{F}_q$ be the finite field of q elements. Let $f, g \in \mathbb{F}_q[X, Y]$. The graph $G_q(f, g)$ is a bipartite graph with vertex partitions $P = \mathbb{F}_q^3$ and $L = \mathbb{F}_q^3$, and edges defined as follows: a vertex $(p) = (p_1, p_2, p_3) \in P$ is adjacent to a vertex $[l] = [l_1, l_2, l_3] \in L$ if and only if $p_2 + l_2 = f(p_1, l_1)$ and $p_3 + l_3 = g(p_1, l_1)$. If $f = XY$ and $g = XY^2$, the graph $G_q(XY, XY^2)$ contains no cycles of length less than eight and is edge-transitive. Motivated by certain questions in extremal graph theory and finite geometry, people search for examples of graphs $G_q(f, g)$ containing no cycles of length less than eight and not isomorphic to the graph $G_q(XY, XY^2)$, even without requiring them to be edge-transitive. So far, no such graphs $G_q(f, g)$ have been found. It was conjectured that if both f and g are monomials, then no such graphs exist. In this paper we prove the conjecture.

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1. Introduction

All graphs considered in this paper are finite, undirected, with no loops or multiple edges. All definitions of graph-theoretic terms that we omit can be found in Bollobás [1]. The *order* of a graph is the number of its vertices. The *degree* of a vertex of a graph is the number of vertices adjacent to it. A graph is called *r-regular* if degrees of all its vertices are equal to r. A graph is called *connected* if every pair of its distinct vertices is connected by a path. The *distance* between two distinct vertices in a connected graph is the length of the shortest path connecting them. The *girth* of a graph containing cycles is the length of a shortest cycle.

Let $k \geq 2$, and $g_k(n)$ denote the greatest number of edges in a graph of order n and girth at least $2k+1$. The function $g_k(n)$ has been studied extensively; see the surveys by Bondy [3], and by Füredi and Simonovits [6]. It is known that for $2 \leq k \neq 5$, and sufficiently large n,

$$c'_k n^{1+\frac{2}{3k-3+\epsilon}} \leq g_k(n) \leq c_k n^{1+\frac{1}{k}}, \tag{1.1}$$

where $\epsilon = 0$ if k is odd, $\epsilon = 1$ if k is even, and c'_k and c_k are positive constants depending on k only. The upper bound is due to Bondy and Simonovits [2], and the lower bound was obtained via an explicit construction by Lazebnik, Ustimenko and Woldar [10]. (For many prior related results see the references in [2,10].) For $k = 5$, a better lower bound is known, and it is of magnitude $n^{1+1/5}$. The only known values of k for which the lower bound for $g_k(n)$ is of (maximum) magnitude $n^{1+1/k}$ are $k = 2, 3$, and 5. Several graphs of such extremal magnitude were constructed using polynomials over finite fields as we describe below.

Let q be a prime power, and let $\mathbb{F}_q$ be the finite field with q elements. For each $k = 2, 3, 5$, consider a bipartite graph $\Gamma_k(q)$ with vertex partitions $P_k = \mathbb{F}_q^k$ and $L_k = \mathbb{F}_q^k$, and edges defined as follows.

For $k = 2$, a vertex $(p) = (p_1, p_2) \in P_2$ is adjacent to a vertex $[l] = [l_1, l_2] \in L_2$ if and only if

$$p_2 + l_2 = p_1 l_1.$$

For $k = 3$, a vertex $(p) = (p_1, p_2, p_3) \in P_3$ is adjacent to a vertex $[l] = [l_1, l_2, l_3] \in L_3$ if and only if the following two equalities hold:

$$p_2 + l_2 = p_1 l_1, \quad p_3 + l_3 = p_1 l_1^2.$$

For $k = 5$, a vertex $(p) = (p_1, p_2, p_3, p_4, p_5) \in P_5$ is adjacent to a vertex $[l] = [l_1, l_2, l_3, l_4, l_5] \in L_5$ if and only if the following four equalities hold:

$$p_2 + l_2 = p_1 l_1, \quad p_3 + l_3 = p_1 l_1^2, \quad p_4 + l_4 = p_1 l_1^3, \quad p_5 + l_5 = p_4 l_1 - 2p_3 l_2 + p_2 l_3.$$