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On the additive energy of the distance set in finite fields

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A B S T R A C T

We use character sums to derive new bounds on the additive energy of the set of distances (counted with multiplicities) between two subsets of a vector space over a given finite field. We also give applications to sumsets of distance sets.

1. Introduction

1.1. Motivation and previous results

Let $\mathbb{F}_q$ be the finite field of q elements. We define the *distance* between two vectors

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$$\mathbf{x} = (x_1, \dots, x_n), \ \mathbf{y} = (y_1, \dots, y_n) \in \mathbb{F}_q^n$$

as

$$d(\mathbf{x}, \mathbf{y}) = \sum_{j=1}^{n} (x_j - y_j)^2.$$

The interest to this function has been motivated by the results of Iosevich and Rudnev [7] on the *Erdős distance problem*, see also [4] for a systematic introduction and [2,5,6,8,11] for various generalisations. We also refer to recent results of Dietmann [3] and Koh and Sun [12] for the state of art on the Erdős distance problem and further references. For example, let

$$\mathcal{D}(\mathcal{X}, \mathcal{Y}) = \# \{d(\mathbf{x}, \mathbf{y}) \ : \ (\mathbf{x}, \mathbf{y}) \in \mathcal{X} \times \mathcal{Y}\}$$

be the distance set of two sets $\mathcal{X}, \mathcal{Y} \subseteq \mathbb{F}_q^n$ of cardinalities $\#\mathcal{X} = X$ and $\#\mathcal{Y} = Y$. If $n \geq 3$ is odd then by [12, Theorem 3.3], we have

$$\#\mathcal{D}(\mathcal{X}, \mathcal{Y}) \geq \begin{cases} \min\left\{\frac{q}{2}, \frac{XY}{8q^{n-1}}\right\} & \text{if } X < q^{(n-1)/2}, \\ \min\left\{\frac{q}{2}, \frac{Y}{8q^{(n-1)/2}}\right\} & \text{if } q^{(n-1)/2} \leq X < q^{(n+1)/2}, \\ \min\left\{\frac{q}{2}, \frac{XY}{8q^{n}}\right\} & \text{if } q^{(n+1)/2} \leq X. \end{cases} \tag{1}$$

Furthermore, if $n \geq 2$ is even and $XY \geq 16q^n$, by [12, Theorem 3.5], we have

$$\#\mathcal{D}(\mathcal{X}, \mathcal{Y}) \geq \begin{cases} \frac{q}{144} & \text{if } X < q^{(n-1)/2}, \\ \min\left\{\frac{q}{144}, \frac{Y}{2q^{(n-1)/2}}\right\} & \text{if } q^{(n-1)/2} \leq X < q^{(n+1)/2}, \\ \min\left\{\frac{q}{144}, \frac{2XY}{q^{n}}\right\} & \text{if } q^{(n+1)/2} \leq X. \end{cases} \tag{2}$$

Here we consider more properties of this set of distances. In particular, given two sets $\mathcal{X}, \mathcal{Y} \subseteq \mathbb{F}_q^n$, we also consider the *additive energy* of the set of distances, counted with multiplicities, that is,

$$\begin{aligned} E_+(\mathcal{X}, \mathcal{Y}) = \#\big\{(\mathbf{x}_i, \mathbf{y}_i)_{i=1}^4 \in (\mathcal{X} \times \mathcal{Y})^4 \ \ : \\ d(\mathbf{x}_1, \mathbf{y}_1) + d(\mathbf{x}_2, \mathbf{y}_2) = d(\mathbf{x}_3, \mathbf{y}_3) + d(\mathbf{x}_4, \mathbf{y}_4)\big\}. \end{aligned}$$

We recall the additive energy of sets is closely related to their combinatorial properties, see [22] for a systematic background and also [16–19] and references therein for more recent results.

Furthermore, some additive character sums can also be estimated via the additive energy. For example, by [13, Lemma 4] (taken with $\ell = m = 2$), for a nontrivial additive