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On a conjecture of Wan about limiting Newton polygons

Yi Ouyang, Jinbang Yang *

Wu Wen-Tsun Key Laboratory of Mathematics, School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui 230026, PR China

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We show that for a monic polynomial $f(x)$ over a number field K containing a global permutation polynomial of degree > 1 as its composition factor, the Newton Polygon of $f \mod \mathfrak{p}$ does not converge for $\mathfrak{p}$ passing through all finite places of K. In the rational number field case, our result is the "only if" part of a conjecture of Wan about limiting Newton polygons.

1. Introduction and main results

Let K be a number field and $f(x)$ be a monic polynomial in $K[x]$ of degree $d \geq 1$. For a finite place $\mathfrak{p}$ of K, denote the completion of K at $\mathfrak{p}$ by $K_{\mathfrak{p}}$. Let $\mathcal{O}_{\mathfrak{p}}$ be the ring of $\mathfrak{p}$-adic integers and $k_{\mathfrak{p}}$ be the residue field. Then $k_{\mathfrak{p}}$ is a finite field of $q = q_{\mathfrak{p}} = p^h$

* Corresponding author.
E-mail addresses: yiouyang@ustc.edu.cn (Y. Ouyang), yjb@mail.ustc.edu.cn (J. Yang).

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elements for some rational prime $p = p_{\mathfrak{p}}$ and some positive integer $h = h_{\mathfrak{p}}$. Denote by $k_{\mathfrak{p}}^m$ the unique field extension of $k_{\mathfrak{p}}$ of degree m. Denote by $\Sigma_K := \Sigma_K(f)$ the set of finite places $\mathfrak{p}$ of K such that $f(x) \in \mathcal{O}_{\mathfrak{p}}[x]$ and $(d,p) = 1$. Note that almost all finite places of K are contained in Σ_K.

Let $\mathfrak{p}$ be a place in Σ_K. By modulo $\mathfrak{p}$, we get the reduction $\overline{f}$ of f, a polynomial over $k_{\mathfrak{p}}$. For a nontrivial character $\chi : \mathbb{F}_p \to \mu_p$, the L-function

$$L(\overline{f}, \chi, t) = L(\overline{f}/k_{\mathfrak{p}}, \chi, t) = \exp\left(\sum_{m=1}^{\infty} S_m(\overline{f}, \chi)\frac{t^m}{m}\right), \tag{1.1}$$

where $S_m(\overline{f}, \chi)$ is the exponential sum

$$S_m(\overline{f}, \chi) = S_m(\overline{f}/k_{\mathfrak{p}}, \chi) = \sum_{x \in k_{\mathfrak{p}}^m} \chi(\mathrm{Tr}_{k_{\mathfrak{p}}^m/\mathbb{F}_p}(\overline{f}(x))), \tag{1.2}$$

is a polynomial of t of degree $d-1$ over $\mathbb{Q}_p(\zeta_p)$ by well-known theorems of Dwork–Bombieri–Grothendieck and Adolphson–Sperber [1]. The q-adic Newton polygon $\mathrm{NP}_{\mathfrak{p}}(f)$ of this L-function does not depend on the choice of the nontrivial character χ.

Let $\mathrm{HP}(f)$ be a convex polygon with break points

$$\left\{ \left(i, \frac{i(i+1)}{2d}\right) \middle| 0 \le i \le d. \right\},$$

which only depends on the degree of f. Adolphson and Sperber [2] proved that $\mathrm{NP}_{\mathfrak{p}}(f)$ lies above $\mathrm{HP}(f)$ and that $\mathrm{NP}_{\mathfrak{p}}(f) = \mathrm{HP}(f)$ if $p \equiv 1 \mod d$. Obviously, there are infinitely many $\mathfrak{p} \in \Sigma_K$ such that $p \equiv 1 \mod d$, thus if $\lim\limits_{\mathfrak{p} \in \Sigma_K} \mathrm{NP}_{\mathfrak{p}}(f)$ exists, then $\lim\limits_{\mathfrak{p} \in \Sigma_K} \mathrm{NP}_{\mathfrak{p}}(f) = \mathrm{HP}(f)$.

Recall that a global permutation polynomial (GPP) over K is a polynomial $P(x) \in K[x]$ such that $x \mapsto \overline{P}(x)$, where $\overline{P}$ is the reduction of P modulo $\mathfrak{p}$, is a permutation on $k_{\mathfrak{p}}$ for infinitely many places $\mathfrak{p} \in \Sigma_K$.

In 1999, D. Wan proposed a conjecture, whose complete version in [16, Chapter 5] and [4, Conjecture 6.1] is as follows:

Conjecture 1.1 *(Wan). Let f be a non-constant monic polynomial in $\mathbb{Q}[x]$. Then f contains a GPP over $\mathbb{Q}$ of degree > 1 as its composition factor if and only if $\lim\limits_{\mathfrak{p} \in \Sigma_{\mathbb{Q}}} \mathrm{NP}_{\mathfrak{p}}(f)$ does not exist.*

In this note, we give a proof of the "only if" part of Wan's conjecture. Moreover, we get the following main result.

Theorem 1.2. *Let f be a non-constant monic polynomial in $K[x]$. If f contains a GPP over K of degree > 1 as its composition factor, then $\lim\limits_{\mathfrak{p} \in \Sigma_K} \mathrm{NP}_{\mathfrak{p}}(f)$ does not exist.*