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Hyperquadratic continued fractions and automatic sequences

Alain Lasjaunias [a,*], Jia-Yan Yao [b]

[a] *Institut de Mathématiques de Bordeaux, CNRS-UMR 5251, Talence 33405, France*
[b] *Department of Mathematics, Tsinghua University, Beijing 100084, People's Republic of China*

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ABSTRACT

We show that three different families of hyperquadratic elements, studied in the literature, have the following property: For these elements, the leading coefficients of the partial quotients in their continued fraction expansion form 2-automatic sequences. We also show that this is not true for algebraic elements in $\mathbb{F}(q)$ in general. Indeed, we use an element studied by Mills and Robbins as counterexample. This element is algebraic in $\mathbb{F}(3)$ but the principal coefficients of the partial quotients do not form an automatic sequence.

* Corresponding author.
E-mail addresses: Alain.Lasjaunias@math.u-bordeaux1.fr (A. Lasjaunias), jyyao@math.tsinghua.edu.cn (J.-Y. Yao).

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1. Introduction

Let $\mathbb{F}_q$ be the finite field containing q elements, with $q = p^s$ where p is a prime number and $s \geqslant 1$ is an integer. We consider the field of power series in $1/T$, with coefficients in $\mathbb{F}_q$, where T is a formal indeterminate. We will denote this field by $\mathbb{F}(q)$. Hence a non-zero element of $\mathbb{F}(q)$ is written as $\alpha = \sum_{k \leqslant k_0} c_k T^k$ with $k_0 \in \mathbb{Z}$, $c_k \in \mathbb{F}_q$, and $c_{k_0} \neq 0$. Noting the analogy of this expansion with a decimal expansion for a real number, it is natural to regard the elements of $\mathbb{F}(q)$ as (formal) numbers and indeed they are analogue to real numbers in many ways.

It is well known that the sequence of coefficients (or digits) $(c_k)_{k \leqslant k_0}$ for α is ultimately periodic if and only if α is rational, that is α belongs to $\mathbb{F}_q(T)$. However, and this is a singularity of the formal case, this sequence of digits can also be characterized for all elements in $\mathbb{F}(q)$ which are algebraic over $\mathbb{F}_q(T)$. The origin of the following theorem can be found in the work of Christol [8] (see also the article of Christol, Kamae, Mendès France, and Rauzy [9]).

Theorem 1 *(Christol). Let α in $\mathbb{F}(q)$ with $q = p^s$. Let $(c_k)_{k \leqslant k_0}$ be the sequence of digits of α and $u(n) = c_{-n}$ for all integers $n \geqslant 0$. Then α is algebraic over $\mathbb{F}_q(T)$ if and only if the following set of subsequences of $(u(n))_{n \geqslant 0}$*

$$K(u) = \left\{ (u(p^i n + j))_{n \geqslant 0} \mid i \geqslant 0,\, 0 \leqslant j < p^i \right\}$$

is finite.

The sequences having the finiteness property stated in this theorem were first introduced in the 1960s by computer scientists. Considered in a larger setting (see the beginning of Section 3), they are now called automatic sequences, and form a class of deterministic sequences which can be defined in several different ways. A full account on this topic and a very complete list of references are to be found in the book of Allouche and Shallit [2]. In this note we want to show a different type of connection between automatic sequences and some particular algebraic power series in $\mathbb{F}(q)$.

Firstly, let us describe these particular algebraic elements. Let α be irrational in $\mathbb{F}(q)$. We say that α is hyperquadratic, if there exists $r = p^t$ with $t \geqslant 0$ an integer such that the elements α^{r+1}, α^r, α, and 1 are linked over $\mathbb{F}_q(T)$. Thus an hyperquadratic element is algebraic over $\mathbb{F}_q(T)$ of degree $\leqslant r + 1$, and the reader may consult [6] where a precise definition was introduced. The subset of hyperquadratic elements in $\mathbb{F}(q)$ is denoted by $\mathcal{H}(q)$. Note that this subset contains the quadratic power series (take $r = 1$) and also the cubic power series (take $r = p$). Originally, these algebraic elements were introduced in the 1970s by Baum and Sweet (see [4]), in the particular case $q = 2$, and later considered in the 1980s by Mills and Robbins [21] and Voloch [25], in all characteristic. It appears that $\mathcal{H}(q)$ contains elements having an arbitrary large algebraic degree. But hyperquadratic power series are rare: an algebraic power series of high algebraic degree