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On the 2-ranks of a class of unitals

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ABSTRACT

Let $\mathcal{U}_\theta$ be a unital defined in a shift plane of odd order q^2, which are constructed recently in [40]. In particular, when the shift plane is desarguesian, $\mathcal{U}_\theta$ is a special Buekenhout–Metz unital formed by a union of ovals. We investigate the dimensions of the binary codes derived from $\mathcal{U}_\theta$. By using Kloosterman sums, we obtain a new lower bound on the aforementioned dimensions which improves Leung and Xiang's result [32,33]. In particular, for $q = 3^m$, this new lower bound equals $\frac{2}{3}(q^3 + q^2 - 2q) - 1$ for even m and $\frac{2}{3}(q^3 + q^2 + q) - 1$ for odd m.

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1. Introduction

Let t, v, k and λ be positive integers. A t-(v, k, λ) *design* $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is a set $\mathcal{P}$ of v points together with a set $\mathcal{B}$ of k-subsets of $\mathcal{P}$ (called blocks) satisfying that every t-subset of $\mathcal{P}$ is contained in exactly λ blocks.

Let m be an integer larger than or equal to 3. A *unital* of *order* m is a 2-$(m^3 + 1, m + 1, 1)$ design.

Most of the known unitals can be embedded in projective planes Π of order q^2. In these cases, a unital is a set $\mathcal{U}$ of $q^3 + 1$ points such that each line of Π intersects $\mathcal{U}$ in 1 or $q + 1$ points. When Π is the desarguesian projective plane $\mathrm{PG}(2, q^2)$, the set of absolute points of a unitary polarity, or equivalently speaking, the rational points on a nondegenerate Hermitian curve form a *classical* unital. There are also non-classical unitals in $\mathrm{PG}(2, q^2)$, for instance Buekenhout–Metz unitals [11] are a proper generalization of the classical ones. There are unitals which are not embeddable in a projective plane, for instance the Ree unitals [35]. Moreover, it is not necessary that the order of a unital is a prime power, for instance, the order of the unitals discovered in [6] equals 6.

Unitals also exist in non-desarguesian planes. For instance, there are unitals derived from unitary polarities in various translation planes and shift planes; see [1,3,15,16,25,28]. Commutative semifield planes, as a special type of translation and shift planes, also contain some unitals which are analogous to the Buekenhout–Metz ones in the desarguesian case [2,45].

Recently in [40], the authors have been investigating the existence and properties of a special type of unitals $\mathcal{U}_\theta$ consisting of the union of ovals in shift planes $\Pi(f)$ of odd order in terms of planar functions f on $\mathbb{F}_{q^2}$. In particular, when the planar function is $f(x) = x^2$, the shift plane $\Pi(f)$ is desarguesian and the unital $\mathcal{U}_\theta$ is exactly the one independently discovered by Hirschfeld and Szönyi [22] and by Baker and Ebert [7], which forms a special type of Buekenhout–Metz unitals in desarguesian planes; see [40] or Section 2 for more details.

Let $\mathcal{D}$ be a design with v points. Fix an arbitrary order on the points of $\mathcal{D}$, say $P_1, \ldots, P_v$. The *characteristic vector* v_B of a block B is the binary vector of length v such that $(v^B)_i = 1$ if $P_i \in B$, and $(v^B)_i = 0$ otherwise. Generally, a *linear code* is an arbitrary subspace of a vector space over a field. For any prime number p, the linear p-ary code defined by $\mathcal{D}$ is the vector space $C_p(\mathcal{D})$ spanned by the characteristic vectors of the blocks of $\mathcal{D}$ over $\mathbb{F}_p$. In another word, $C_p(\mathcal{D})$ is the subspace generated by the rows of the incidence matrix of $\mathcal{D}$ over $\mathbb{F}_p$. The *p-rank* of $\mathcal{D}$ is defined as the dimension of $C_p(\mathcal{D})$. It is well-known that $C_p(\mathcal{D})$ is of interest only if p divides the order of $\mathcal{D}$; see [5, Theorem 2.4.1].

The p-ranks of designs have been studied intensively for several reasons. First, if two designs are isomorphic, then the p-ranks of them are the same. Hence, the p-ranks can help us to distinguish two non-isomorphic designs. Second, there are a few long-standing open problems related to p-ranks. For instance, Hamada [21] conjectured that the p-rank of a design with the same parameters of a geometric design $\mathrm{PG}_d(n, q)$ or $\mathrm{AG}_d(n, q)$ is