Finite Fields and Their Applications 39 (2016) 96–110

Contents lists available at ScienceDirect

Finite Fields and Their Applications

www.elsevier.com/locate/ffa

Direct products in projective Segre codes ☆

Azucena Tochimani [a], Maria Vaz Pinto [b], Rafael H. Villarreal [a,*]

[a] *Departamento de Matemáticas, Centro de Investigación y de Estudios Avanzados del IPN, Apartado Postal 14–740, 07000 Mexico City, D.F., Mexico*
[b] *Departamento de Matemática, Instituto Superior Técnico, Universidade Técnica de Lisboa, Avenida Rovisco Pais, 1, 1049-001 Lisboa, Portugal*

A R T I C L E I N F O

Article history:
Received 5 March 2015
Received in revised form 20 September 2015
Accepted 12 January 2016
Available online 25 January 2016
Communicated by L. Storme

MSC:
primary 13P25
secondary 15A78, 14G50, 11T71, 94B27, 94B05

Keywords:
Reed–Muller code
Segre code
Direct product
Segre embedding
Finite field
Minimum distance
Hilbert function
Tensor product

A B S T R A C T

Let $K = \mathbb{F}_q$ be a finite field. We introduce a family of projective Reed–Muller-type codes called *projective Segre codes*. Using commutative algebra and linear algebra methods, we study their basic parameters and show that they are direct products of projective Reed–Muller-type codes. As a consequence we recover some results on projective Reed–Muller-type codes over the Segre variety and over projective tori.

☆ The first author was partially supported by a CONACyT scholarship. The second author is a member of the Center for Mathematical Analysis, Geometry, and Dynamical Systems, Departamento de Matematica, Instituto Superior Tecnico, 1049-001 Lisboa, Portugal. The third author was partially supported by SNI.

* Corresponding author.
E-mail addresses: tochimani@math.cinvestav.mx (A. Tochimani), vazpinto@math.ist.utl.pt (M. Vaz Pinto), vila@math.cinvestav.mx (R.H. Villarreal).

http://dx.doi.org/10.1016/j.ffa.2016.01.006

1. Introduction

Reed–Muller-type evaluation codes have been extensively studied using commutative algebra methods (e.g., Hilbert functions, resolutions, Gröbner bases); see [3,10,27] and the references therein. In this paper we use these methods—together with linear algebra techniques—to study projective Segre codes over finite fields.

Let K be an arbitrary field, let a_1, a_2 be two positive integers, let $\mathbb{P}^{a_1-1}$, $\mathbb{P}^{a_2-1}$ be projective spaces over K, and let $K[\mathbf{x}] = K[x_1, \ldots, x_{a_1}]$, $K[\mathbf{y}] = K[y_1, \ldots, y_{a_2}]$, $K[\mathbf{t}] = K[t_{1,1}, \ldots, t_{a_1,a_2}]$ be polynomial rings with the standard grading. If $d \in \mathbb{N}$, let $K[\mathbf{t}]_d$ denote the set of homogeneous polynomials of total degree d in $K[\mathbf{t}]$, together with the zero polynomial. Thus $K[\mathbf{t}]_d$ is a K-linear space and $K[\mathbf{t}] = \oplus_{d=0}^{\infty} K[\mathbf{t}]_d$. In this grading each $t_{i,j}$ is homogeneous of degree one.

Given $\mathbb{X}_i \subset \mathbb{P}^{a_i-1}$, $i = 1, 2$, denote by $I(\mathbb{X}_1)$ (resp. $I(\mathbb{X}_2)$) the *vanishing ideal* of $\mathbb{X}_1$ (resp. $\mathbb{X}_2$) generated by the homogeneous polynomials of $K[\mathbf{x}]$ (resp. $K[\mathbf{y}]$) that vanish at all points of $\mathbb{X}_1$ (resp. $\mathbb{X}_2$). The *Segre embedding* is given by

$$\psi\colon \mathbb{P}^{a_1-1} \times \mathbb{P}^{a_2-1} \to \mathbb{P}^{a_1 a_2 - 1}$$
$$([(\alpha_1, \ldots, \alpha_{a_1})], [(\beta_1, \ldots, \beta_{a_2})]) \to [(\alpha_i \beta_j)],$$

where $[(\alpha_i\beta_j)] := [(\alpha_1\beta_1, \alpha_1\beta_2, \ldots, \alpha_1\beta_{a_2}, \ldots, \alpha_{a_1}\beta_1, \alpha_{a_1}\beta_2, \ldots, \alpha_{a_1}\beta_{a_2})]$. The map ψ is well-defined and injective [20, p. 13]. The image of $\mathbb{X}_1 \times \mathbb{X}_2$ under the map ψ, denoted by $\mathbb{X}$, is called the *Segre product* of $\mathbb{X}_1$ and $\mathbb{X}_2$. The vanishing ideal $I(\mathbb{X})$ of $\mathbb{X}$ is a graded ideal of $K[\mathbf{t}]$, where the $t_{i,j}$ variables are ordered as $t_{1,1}, \ldots, t_{1,a_2}, \ldots, t_{a_1,1}, \ldots, t_{a_1,a_2}$. The Segre embedding is used in algebraic geometry, among other applications, to show that the product of projective varieties is again a projective variety, see [19, Lecture 2]. If $\mathbb{X}_i = \mathbb{P}^{a_i-1}$ for $i = 1, 2$, the set $\mathbb{X}$ is a projective variety and is called a *Segre variety* [19, p. 25]. The Segre embedding is used in coding theory, among other applications, to study the generalized Hamming weights of some product codes; see [29] and the references therein.

The contents of this paper are as follows. Let $K = \mathbb{F}_q$ be a finite field. In Section 2 we recall two results about the basic parameters and the second generalized Hamming weight of direct product codes (see Theorems 2.1 and 2.2). Then we introduce the family of projective Reed–Muller-type codes, examine their basic parameters, and explain the relation between Hilbert functions and projective Reed–Muller-type codes (see Proposition 2.7). For an arbitrary field K we show that $K[\mathbf{t}]/I(\mathbb{X})$ is the Segre product of $K[\mathbf{x}]/I(\mathbb{X}_1)$ and $K[\mathbf{y}]/I(\mathbb{X}_2)$ (see Definition 2.8 and Theorem 2.10). The Segre product is a subalgebra of

$$(K[\mathbf{x}]/I(\mathbb{X}_1)) \otimes_K (K[\mathbf{y}]/I(\mathbb{X}_2)),$$

the tensor product algebra. Segre products have been studied by many authors; see [9, 18,21] and the references therein. We give full proofs of two results for which we could