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A finite embedding theorem for partial Steiner 3-designs ☆

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A B S T R A C T

A Steiner system $S(t,k,n)$ is a k-uniform set system on $[n]$ for which every t-set is covered exactly once. More generally, a partial Steiner system $P(t,k,n)$ is a k-uniform set system on $[n]$ where every t-set is covered at most once. Let q be a prime power. Using circle geometries and field-based block spreading, we give an explicit embedding for any partial Steiner system $P(3,q+1,n)$ into a Steiner system $S(3,q+1,q^m+1)$ for some $m=m(q,n)$.

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1. Introduction

Let $t \leq k \leq n$ be nonnegative integers. A *partial Steiner system* $P(t, k, n)$ is an n-element set of points equipped with a family $\mathcal{B}$ of k-element subsets, called *blocks*, such that every t-element subset of points is contained in at most one block. We may assume that the ground set is $[n] := \{1, \dots, n\}$ so that $\mathcal{B} \subseteq \binom{[n]}{k}$, the set of all k-subsets of $[n]$.

In a $P(t, k, n)$, a t-set $T \in \binom{[n]}{t}$ is *covered by* $\mathcal{B}$ (or simply *covered*) if there exists a block $B \in \mathcal{B}$ with $B \supseteq T$. When context is clear, we use $\mathcal{T}$ to denote the t-sets covered by $\mathcal{B}$. If every t-set is covered – that is, if $\mathcal{T} = \binom{[n]}{t}$ – we obtain a *Steiner system*. Steiner systems with $t = 1$ are simply uniform partitions. The case $t = 2$ is connected with (pairwise balanced) block designs and finite geometries, and already there are some deep questions on the existence of $S(2, k, n)$. For Steiner systems with general t, there is an exciting preprint by Peter Keevash [11] which announces a randomized algebraic construction. Until this, very little had been known for $t > 2$. Indeed, the only nontrivial infinite families explicitly known have $t = 3$ and $k = q + 1$, q a prime power. These are seeded by Witt's classical 'circle geometries' [22].

Fact 1.1. For any prime power $q > 1$, there is an $S(3, q + 1, q^d + 1)$ for all $d \geq 1$.

Such a construction is furnished by the action of $\mathrm{PGL}(2, q^d)$ on a copy of $\mathbb{F}_q \cup \{\infty\}$ in its natural containment in $\mathbb{F}_{q^d} \cup \{\infty\}$. With the exception of the new asymptotic results in [11], there are no known nontrivial $S(3, k, n)$ for $k - 1$ not a prime power. The same goes for nontrivial $S(t, k, n)$ for $t > 5$.

A partial Steiner system $\mathcal{B}$ on $[n]$ is a *subsystem* of or is *embedded* in another $\mathcal{C}$ on $[m]$ if $n \leq m$ and $\mathcal{B} \subseteq \mathcal{C}$. A natural question is whether a given $P(t, k, n)$ can be embedded in some $S(t, k, m)$. Even for $t = 2$, this problem has a rich history. In 1971, Treash [17] obtained a general embedding for a partial Steiner triple system $P(2, 3, n)$ in a Steiner triple system $S(2, 3, m)$ for some $m < 4^n$. In 1977, Lindner [12] conjectured that any partial Steiner triple system of order n has an embedding in a Steiner triple system of order m if $m \equiv 1, 3 \pmod 6$ and $m \geq 2n + 1$. The congruence is necessary and the inequality is sharp; these come from basic counting. Recently Bryant and Horsley [4] settled Lindner's conjecture in the affirmative.

Ganter [6] and Quackenbush [16] discussed the finite embedding problem for partial Steiner systems $P(2, k, n)$ with $k = q+1$ and q, respectively, for q a prime power. Finally, Ganter [7] showed that any partial Steiner system $P(2, k, n)$ can be embedded in a Steiner system $S(2, k, m)$. This work has been refined and extended in various ways. Wilson's asymptotic theory [20] on graph decompositions is a generalization. Vu [18] established a lower bound on the number of non-isomorphic such embeddings. Colbourn et al. [5] considered embeddings of more general block designs.