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A characterization of translation ovals in finite even order planes

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ABSTRACT

In this article we consider a set $\mathcal{C}$ of points in $\mathrm{PG}(4, q)$, q even, satisfying certain combinatorial properties with respect to the planes of $\mathrm{PG}(4, q)$. We show that there is a regular spread in the hyperplane at infinity, such that in the corresponding Bruck–Bose plane $\mathrm{PG}(2, q^2)$, the points corresponding to $\mathcal{C}$ form a translation hyperoval, and conversely.

1. Introduction

In this article we first consider a non-degenerate conic in $\mathrm{PG}(2, q^2)$, q even. We look at the corresponding point set in the Bruck–Bose representation in $\mathrm{PG}(4, q)$, and study its combinatorial properties (details of the Bruck–Bose representation are given in Section 2). Some properties of this set were investigated in [4]. In this article we are interested

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in combinatorial properties relating to planes of $\mathrm{PG}(4,q)$. We consider a set of points in $\mathrm{PG}(4,q)$ satisfying certain of these combinatorial properties and find that the points correspond to a translation oval in the Bruck–Bose plane $\mathrm{PG}(2,q^2)$.

In [3], the case when q is odd is considered, and we show that given a set of points in $\mathrm{PG}(4,q)$ satisfying the following combinatorial properties, we can reconstruct the conic in $\mathrm{PG}(2,q^2)$. We use the following terminology in $\mathrm{PG}(4,q)$: if the hyperplane at infinity is denoted Σ_∞, then we call the points of $\mathrm{PG}(4,q)\setminus\Sigma_\infty$ *affine points.*

Theorem 1.1. *(See [3].) Let Σ_∞ be the hyperplane at infinity in $\mathrm{PG}(4,q)$, $q\ge 7$, q odd. Let $\mathcal{C}$ be a set of q^2 affine points, called $\mathcal{C}$-points, and suppose there exists a set of planes called* $\mathcal{C}$-planes *satisfying the following properties:*

1. *Each $\mathcal{C}$-plane meets $\mathcal{C}$ in a q-arc.*
2. *Any two distinct $\mathcal{C}$-points lie in a unique $\mathcal{C}$-plane.*
3. *The affine points of $\mathrm{PG}(4,q)$ are of three types: points of $\mathcal{C}$; points on no $\mathcal{C}$-plane; and points on exactly two $\mathcal{C}$-planes.*
4. *If a plane meets $\mathcal{C}$ in more than four points, it is a $\mathcal{C}$-plane.*

Then there exists a unique spread $\mathcal{S}$ in Σ_∞ so that in the Bruck–Bose translation plane $\mathcal{P}(\mathcal{S})$, the $\mathcal{C}$-points form a q^2-arc of $\mathcal{P}(\mathcal{S})$. Moreover, the spread $\mathcal{S}$ is regular, and so $\mathcal{P}(\mathcal{S})\cong\mathrm{PG}(2,q^2)$, and the q^2-arc can be completed to a conic of $\mathrm{PG}(2,q^2)$.

The case when q is even is more complex. The combinatorial properties only allow us to reconstruct a translation oval in $\mathrm{PG}(2,q^2)$. The main result of this article is the following theorem.

Theorem 1.2. *Consider $\mathrm{PG}(4,q)$, q even, $q>2$, with the hyperplane at infinity denoted by Σ_∞. Let $\mathcal{C}$ be a set of q^2 affine points, called* $\mathcal{C}$-points *and consider a set of planes called* $\mathcal{C}$-planes *which satisfies the following:*

(A1) *Each $\mathcal{C}$-plane meets $\mathcal{C}$ in a q-arc.*
(A2) *Any two distinct $\mathcal{C}$-points lie in a unique $\mathcal{C}$-plane.*
(A3) *The affine points that are not in $\mathcal{C}$ lie on exactly one $\mathcal{C}$-plane.*
(A4) *Every plane which meets $\mathcal{C}$ in at least three points either meets $\mathcal{C}$ in exactly four points or is a $\mathcal{C}$-plane.*

Then there exists a regular spread $\mathcal{S}$ in Σ_∞ such that in the Bruck–Bose plane $\mathcal{P}(\mathcal{S})\cong\mathrm{PG}(2,q^2)$, the $\mathcal{C}$-points, together with two extra points on ℓ_∞, form a translation hyperoval of $\mathrm{PG}(2,q^2)$.

We begin in Section 2 with the necessary background material on the Bruck–Bose representation. In Section 3 we investigate combinatorial properties of conics and translation