Finite Fields and Their Applications 33 (2015) 66–79

Contents lists available at ScienceDirect

Finite Fields and Their Applications

www.elsevier.com/locate/ffa

Codes over a weighted torus

Eduardo Dias [a,1], Jorge Neves [b,*,2]

[a] *Mathematics Institute, Zeeman Building, University of Warwick, Coventry CV4 7AL, UK*

[b] *CMUC, Department of Mathematics, University of Coimbra, 3001-454 Coimbra, Portugal*

ARTICLE INFO

Article history:
Received 28 February 2014
Received in revised form 30 October 2014
Accepted 20 November 2014
Available online 4 December 2014
Communicated by W. Cary Huffman

MSC:
primary 13P25
secondary 14G50, 14G15, 11T71, 94B27, 94B05

Keywords:
Projective Reed–Muller codes
Maximum distance separable codes
Vanishing ideals
Numerical semigroups
Hilbert series
Index of regularity

ABSTRACT

We define the notion of weighted projective Reed–Muller codes over a subset $X \subset \mathbb{P}(w_1, \ldots, w_s)$ of a weighted projective space over a finite field. We focus on the case when X is a projective weighted torus. We show that the vanishing ideal of X is a lattice ideal and relate it with the lattice ideal of a minimal presentation of the semigroup algebra of the numerical semigroup $Q = \langle w_1, \ldots, w_s \rangle \subset \mathbb{N}$. We compute the index of regularity of the vanishing ideal of X in terms of the weights of the projective space and the Frobenius number of Q. We compute the basic parameters of weighted projective Reed–Muller codes over a 1-dimensional weighted torus and prove they are maximum distance separable codes.

* Corresponding author.
E-mail addresses: eduardo.m.dias@gmail.com (E. Dias), neves@mat.uc.pt (J. Neves).

[1] The work of the first author was supported by POPH and QREN through FCT grant SFRH/BD/61216/2009.

[2] The second author was partially funded by CMUC, through European program COMPETE/FEDER and FCT project PEst-C/MAT/UI0324/2011. Part of work was developed during a research visit of the second author to CINVESTAV of the IPN, México, under the financial support of a research grant from Santander Totta Bank (Portugal).

http://dx.doi.org/10.1016/j.ffa.2014.11.009
1071-5797/

1. Introduction

A standard projective Reed–Muller code, $C_X(d)$, is the image of the degree d homogeneous component of a standard polynomial ring $K[t_1, \ldots, t_s]$ over a finite field K by a homomorphism defined by evaluation of forms of degree d on the points of an arbitrary subset $X \subset \mathbb{P}^s$. In this work we define the notion of *weighted projective Reed–Muller code* (see Definition 3.1). This notion differs from the standard definition in that the grading of $K[t_1, \ldots, t_s]$, which is given by $\deg(t_i) = w_i \geq 1$, for coprime integers w_i, is not necessarily the standard one. We focus on the family of codes $C_{\mathbb{T}}(d)$ associated with a weighted $(s-1)$-dimensional projective torus $\mathbb{T}(w_1, \ldots, w_s)$ (see Definition 2.4).

Standard projective Reed–Muller codes of order $d \leq q$ were defined and studied by Lachaud in [17,18] and, for all $d \geq 0$, by Sørensen in [31]. Much of the recent research on projective Reed–Muller codes over an arbitrary subset of $X \subset \mathbb{P}^s$ focuses on the computation of their basic parameters: *length*, *dimension* and *minimum distance* (see Definition 3.3). When $X = \mathbb{P}^s$ all the basic parameters are known (*cf.* [18,31]); in particular, projective Reed–Muller codes over $\mathbb{P}^1$ are *maximum distance separable* codes. In general, an approach to this computation using commutative algebra (as in [26]) relies on a good understanding of $I_X \subset K[t_1, \ldots, t_s]$, the vanishing ideal of X. Many authors have studied projective Reed–Muller codes over a subset $X \subset \mathbb{P}^s$ for which the ideal I_X is well understood; *e.g.*, when X is the set of rational points of a complete intersection, *cf.* [1,4,5,7,11,15,29,30], when X is the Segre embedding of the product of two projective spaces, *cf.* [12], when X is the Veronese variety, *cf.* [13], when X is an affine Cartesian product, *cf.* [3,19,10], when X is the projective torus in $\mathbb{P}^s$ and, more generally, when X is an *algebraic toric subset*, *cf.* [21–23,25,28,29].

The advantage of working with subsets of the torus is that for a certain subclass of these subsets (consisting of algebraic toric subsets, as defined by Villarreal et al. in [25,27]) the ideal I_X is a lattice ideal. Like in the standard case, $I_{\mathbb{T}}$, the vanishing ideal of the weighted torus $\mathbb{T}(w_1, \ldots, w_s)$, is also a lattice ideal. Indeed, we show that $I_{\mathbb{T}}$ is Cohen–Macaulay, 1-dimensional and can be obtained from the lattice ideal of a minimal presentation of the semigroup algebra of the numerical semigroup $Q = \langle w_1, \ldots, w_s \rangle \subset \mathbb{N}$ (*cf.* Theorem 2.8). The lattice ideal of a minimal presentation of the semigroup algebra was first studied by Herzog in [16]. He gives a sufficient condition for this ideal to be a complete intersection (see Remark 2.12), which, combined with our results, is also a sufficient condition for $I_{\mathbb{T}}$ to be a complete intersection. The relation between $I_{\mathbb{T}}$ and the lattice ideal of a minimal presentation of Q enables the computation of the Hilbert series and the index of regularity of $K[t_1, \ldots, t_s]/I_{\mathbb{T}}$ in terms of $w_1, \ldots, w_s$ and the Frobenius number of Q (*cf.* Theorem 3.8 and Corollary 3.9). The importance, from a coding theory point of view, of the knowledge of the index of regularity is clearer in the case of standard projective Reed–Muller codes. Here, the function $\dim_K C_X(d)$ is strictly increasing and the value of d for which $\dim_K C_X(d)$ becomes constant and is equal to the dimension of the ambient space (thus, for which $C_X(d)$ becomes the trivial code) is precisely given by the index of regularity. In the weighted case $\dim_K C_X(d)$ is not necessarily an increasing