Finite Fields and Their Applications 37 (2016) 28–35

Contents lists available at ScienceDirect

Finite Fields and Their Applications

www.elsevier.com/locate/ffa

On coefficients of Carlitz cyclotomic polynomials

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ARTICLE INFO

Article history:
Received 24 May 2014
Accepted 31 August 2015
Available online 18 September 2015
Communicated by Gary L. Mullen

MSC:
11R60
11G09
11A99

Keywords:
Carlitz polynomials
Carlitz cyclotomic polynomials
Pillai function

ABSTRACT

Let $n \in \mathbb{Z}_+$, and $\Phi_n(x)$ be the nth classical cyclotomic polynomial. In [4, Theorem 1], D. Lehmer showed that the geometric mean of $\{\Phi_s(1) : s, n \in \mathbb{Z}_+, s \leq n\} \to e \approx 2.71828$, as $n \to \infty$. Replacing $\mathbb{Z}$ by $\mathbb{F}_q[T]$, and the nth elementary cylcotomic polynomial $\Phi_n(x)$ by the Carlitz m-cyclotomic polynomial $\Phi_m(x)$, where $m \in \mathbb{F}_q[T]$, we obtain an analogue to Lehmer's result. We also express $\Phi_m(0) \in \mathbb{F}_2[T]$ in terms of $\phi_*(\cdot)$, the Pillai polynomial function. The resulting expression is a function field analogue of Hölder's formula for $\Phi_n(1)$.

1. Introduction

Let n be a positive integer, the elementary nth cyclotomic polynomial is the polynomial

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[1] This work was carried out with financial support from the Stellenbosch Postgraduate Merit Bursary Scheme and the government of Canada's International Development Research Centre (IDRC) grant number SAMUG2009004S, and within the framework of the AIMS Research for Africa Project.

http://dx.doi.org/10.1016/j.ffa.2015.08.006
1071-5797/

$$\Phi_n(x) = \prod_{j=1}^{n}{}' (x - \zeta_j),$$

where ζ_j are the nth roots of unity in $\mathbb{C}$ given by $\zeta_j = e^{\frac{2\pi i j}{n}}$. The prime in the product simply means that the product is taken over all positive integers less than n that are co-prime to n, i.e., the ζ_j's are the primitive nth roots of unity. The degree of $\Phi_n(x)$ is $\varphi(n)$, where $\varphi(n)$ is the Euler Totient function. Since each nth root of unity is a primitive dth root of unity for a unique d dividing n (corollary to Lagrange's Theorem), we have the following fundamental relation involving cyclotomic polynomials (as d runs over positive divisors of n),

$$x^n - 1 = \prod_{d|n} \Phi_d(x). \tag{1}$$

Applying the (multiplicative form of the) Möbius inversion formula to Equation (1), we get

$$\Phi_n(x) = \prod_{d|n} (x^d - 1)^{\mu(\frac{n}{d})}. \tag{2}$$

Here μ is the Möbius function. Recall that this is the arithmetic function defined as

$$\mu(n) = \begin{cases} (-1)^s, & n \text{ is square free with } t \text{ distinct prime factors,} \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

The Möbius function satisfies the following identity [10, Theorem 2.1]

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & n = 1 \\ 0, & n \neq 1. \end{cases}$$

It is well known that $\Phi_n(x)$ is a monic polynomial, has integer coefficients and is irreducible over $\mathbb{Q}$, see [5]. There is enormous literature concerning properties and coefficients of cyclotomic polynomials, see [7,11,12] and a number of others focusing on the $\mathbb{F}_q[T]$ analogues of these objects, e.g., [2,8,9]. However, in all these resources, the analogues of the Theorems 1.1, 1.2 and 1.3 have not been explored. It is the analogues of these results that we attempt to prove in this article. To achieve this, let us first recall the classical situation.

The von Mangoldt function is an arithmetic function defined as follows:

$$\Lambda(n) := \begin{cases} 0, & n \neq p^s \\ \log(p), & n = p^s. \end{cases} \tag{4}$$

It satisfies the following identities: if n is a positive integer, then