Finite Fields and Their Applications 37 (2016) 72–84

Contents lists available at ScienceDirect

Finite Fields and Their Applications

www.elsevier.com/locate/ffa

Symmetric Bush-type generalized Hadamard matrices and association schemes

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A R T I C L E   I N F O

Article history:
Received 12 June 2015
Received in revised form 11 September 2015
Accepted 13 September 2015
Available online 26 September 2015
Communicated by Dieter Jungnickel

MSC:
05B30
05E30

Keywords:
Hadamard matrix
Generalized Hadamard matrix
Bush-type Hadamard matrix
Association scheme

A B S T R A C T

We define Bush-type generalized Hadamard matrices over abelian groups and construct symmetric Bush-type generalized Hadamard matrices over the additive group of finite field $\mathbb{F}_q$, q a prime power. We then show and study an association scheme obtained from such generalized Hadamard matrices.

1. Introduction

A *Hadamard matrix* H of order n is a square matrix of order n with entries from $\{1, -1\}$ such that $HH^T = nI_n$, where H^T is the transpose of H and I_n is the identity

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http://dx.doi.org/10.1016/j.ffa.2015.09.003

matrix of order n. A Hadamard matrix $H = (H_{ij})_{i,j=1}^{2n}$ of order $4n^2$, where each H_{ij} is a square matrix of order $2n$, is said to be of *Bush-type* if $H_{ii} = J_{2n}$ for any $i \in \{1, \dots, 2n\}$, and $H_{ij}J_{2n} = J_{2n}H_{ij} = 0$ for any distinct $i, j \in \{1, \dots, 2n\}$, where J_{2n} is the square matrix of order $2n$ with all one's entries. Bush-type Hadamard matrices have been studied in [1,3,5,8,9]. In particular, it was shown in [3] that the existence of symmetric Bush-type Hadamard matrices is equivalent to that of certain symmetric association schemes of class 3. Furthermore, the association scheme of class 3 is related to strongly regular graphs with certain parameters with the property that the vertex set is decomposed into maximal cliques attaining Delsarete–Hoffmann's bound [4] (see also [8, Lemma 1.1]).

If $\{1, -1\}$ is regarded as a multiplicative group, then one may consider a *generalized Hadamard matrix* as a matrix with entries in a finite abelian group with a multiplicative property, see Section 2.1.

In this paper, we define Bush-type generalized Hadamard matrices over an abelian group and demonstrate a construction method by using some special generalized Hadamard matrices over the additive group of finite field $\mathbb{F}_q$, q a prime power and Latin squares obtained from the same field. In particular, we focus on the symmetric Bush-type generalized Hadamard matrices of order q^2 over the additive group of $\mathbb{F}_q$, q is a prime power.

We will see that some symmetric association schemes having interesting properties are obtained from these matrices with a linear map from $\mathbb{F}_q$ into a subfield preserving addition and describe the eigenmatrices by the Kloosterman sums. The association scheme can be regarded as an extension of the association schemes of class 3 obtained from symmetric Bush-type Hadamard matrices, and it is shown to be a fission scheme of the scheme for the case $n = 2$ in [2, Theorem 1]. In particular, our scheme is a fission scheme of the Hamming association scheme $H(2, q)$.

For the case where $q = 2^m$ or $q = 3^m$ and a linear map being the absolute trace, the eigenmatrices of the association schemes are given explicitly by calculating the Kloosterman sums. Furthermore, the association scheme for the case $q = 3^m$ is used to provide an affirmative answer to a recent question raised by Leopardi.

2. Preliminaries

2.1. Generalized Hadamard matrices

Let G be an additively written finite abelian group of order g. A square matrix $H = (h_{ij})_{i,j=1}^{g\lambda}$ of order $g\lambda$ with entries from G is called a *generalized Hadamard matrix* with the parameters (g, λ) (or $GH(g, \lambda)$) over G if for all distinct $i, k \in \{1, 2, \dots, g\lambda\}$, the multiset $\{h_{ij} - h_{kj} : 1 \le j \le g\lambda\}$ contains exactly λ times of each element of G. The matrix H is *normalized* if H has the first row consisting of 0, where 0 denotes the identity element of G. Any generalized Hadamard matrix is transformed to be normalized by adding suitable elements to the columns.