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Blocking sets of the classical unital

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ABSTRACT

It is known that the classical unital arising from the Hermitian curve in $\mathrm{PG}(2,9)$ does not have a 2-coloring without monochromatic lines. Here we show that for $q \geq 4$ the Hermitian curve in $\mathrm{PG}(2,q^2)$ does possess 2-colorings without monochromatic lines. We present general constructions and also prove a lower bound on the size of blocking sets in the classical unital.

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1. Introduction

In any point–line geometry (or, much more generally, any hypergraph) a *blocking set* is a subset B of the point set that has nonempty intersection with each line (or each edge).

Blocking sets in the finite Desarguesian projective planes $\mathrm{PG}(2,q)$ have been investigated in great detail [16,17]. Since in a projective plane any two lines meet, every set containing a line is a blocking set. A blocking set of a projective plane is called *non-trivial* or *proper* when it does not contain a line. We shall also call blocking sets in other point–line geometries *proper* when they do not contain a line. By definition the complement of a proper blocking set is again one, and every 2-coloring (vertex coloring with two colors such that no line is monochromatic) provides a complementary pair of proper blocking sets.

A blocking set is *minimal* when no proper subset is a blocking set. A blocking set in $\mathrm{PG}(2,q)$ is *small* when its size is smaller than $3(q+1)/2$.

This latter definition was motivated by the important results of Sziklai and Szőnyi, who proved the following '1 (mod p)' result for small minimal blocking sets B in $\mathrm{PG}(2,q)$.

Theorem 1.1 *(Sziklai and Szőnyi). (See [16,17].) Let B be a small minimal blocking set in $PG(2,q)$, $q=p^h$, p prime, $h \geq 1$. Then B intersects every line in $1 \pmod p$ points.*

If e is the largest integer such that B intersects every line in $1 \pmod{p^e}$ points, then e is a divisor of h, and every line of $PG(2,q)$ that intersects B in exactly $1+p^e$ points intersects B in a subline $PG(1,p^e)$.

A *unital* $\mathcal{U}$ of order q is a 2-$(q^3+1, q+1, 1)$ design, an incidence structure consisting of a set of q^3+1 points and a collection of subsets of size $q+1$ called *blocks*, where any two distinct points are incident with a unique block.

The classical example of a unital of order q arises from a Hermitian curve $\mathcal{H}(2,q^2)$ of $\mathrm{PG}(2,q^2)$. The absolute points of a Hermitian polarity in $\mathrm{PG}(2,q^2)$ constitute an absolutely irreducible curve Γ of degree $q+1$ in $\mathrm{PG}(2,q^2)$, projectively equivalent to the algebraic curve with equation $X^{q+1}+Y^{q+1}+Z^{q+1}=0$. It is known that $|\Gamma| = q^3+1$ in $\mathrm{PG}(2,q^2)$ and every line in $\mathrm{PG}(2,q^2)$ meets Γ in either 1 or $q+1$ points. In the first case, the line is called a *tangent line* to Γ, in the second case a *secant line* to Γ. The blocks of the classical unital are the intersections of the secant lines with Γ.

In this article we investigate blocking sets of the classical unital $\mathcal{U}$ arising from the Hermitian curve $\mathcal{H}(2,q^2)$ of $\mathrm{PG}(2,q^2)$.