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Subgeometries and linear sets on a projective line

Michel Lavrauw *,[1], Corrado Zanella [2]

Dipartimento di Tecnica e Gestione dei Sistemi Industriali, Università di Padova, Stradella S. Nicola 3, 36100 Vicenza, Italy

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A B S T R A C T

We define the splash of a subgeometry on a projective line, extending the definition of [1] to general dimension and prove that a splash is always a linear set. We also prove the converse: each linear set on a projective line is the splash of some subgeometry. Therefore an alternative description of linear sets on a projective line is obtained. We introduce the notion of a club of rank r, generalizing the definition from [4], and show that clubs correspond to tangent splashes. We obtain a condition for a splash to be a scattered linear set and give a characterization of clubs, or equivalently of tangent splashes. We also investigate the equivalence problem for tangent splashes and determine a necessary and sufficient condition for two tangent splashes to be (projectively) equivalent.

1. Introduction and motivation

Given a subgeometry π_0 and a line l_∞ in a projective space π, by extending the hyperplanes of π_0 to hyperplanes of π and intersecting these with the line l_∞, one obtains

* Corresponding author.
E-mail addresses: michel.lavrauw@unipd.it (M. Lavrauw), corrado.zanella@unipd.it (C. Zanella).

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a set of points on the projective line l_∞. Precisely, if we denote the set of hyperplanes of a projective space π by $\mathcal{H}(\pi)$, and $\overline{U}$ denotes the extension of a subspace U of the subgeometry π_0 to a subspace of π, then we obtain the set of points $\{l_\infty \cap \overline{H} : H \in \mathcal{H}(\pi_0)\}$. These sets have been studied in [1] and [2] for Desarguesian planes and cubic extensions, i.e. for a subplane $\pi_0 \cong \mathrm{PG}(2,q)$ in $\pi \cong \mathrm{PG}(2,q^3)$, where such a set is called the *splash of* π_0 *on* l_∞. If l_∞ is tangent (respectively external) to π_0, then a splash is called the *tangent splash* (respectively *external splash*) of π_0 on l_∞. Note that when l_∞ is secant to π_0, the splash of π_0 on l_∞ is just a subline. We study the splash of a subgeometry $\mathrm{PG}(r-1,q)$ in $\mathrm{PG}(r-1,q^n)$ on a line l_∞.

The article is structured as follows. In Section 2 we collect the necessary definitions and notation in order to make the paper self contained and accessible. In Section 3 we show the equivalence between splashes and linear sets on a projective line (Theorem 3.1) and prove that the weight of a point of the linear set is determined by the number of hyperplanes through that point, leading to a characterization of scattered linear sets. In Section 4 we obtain a geometric characterization of so-called clubs or equivalently of tangent splashes, and count the number of distinct tangent splashes in $\mathrm{PG}(1,q^n)$. We conclude with Section 5, where we study the projective equivalence of tangent splashes.

This work is motivated by the link between splashes and linear sets on a projective line. The concept of a splash of a subplane, although quite a natural geometric object to consider, has been studied only recently, see [1,2]. This paper extends the definition of a splash from subplanes to subgeometries of order q in higher dimensional projective spaces, and from cubic to general extension fields. Moreover, this generalization leads to a new interpretation of linear sets on a projective line. The equivalence stated in Theorem 3.1 may turn out useful in investigating linear sets, for instance by linking them to certain ruled surfaces in affine $(2n)$-dimensional spaces over $\mathbb{F}_q$, relying on results from [1,2]. Linear sets and field reduction have played an important role in the construction and characterization of many objects in finite geometry in recent years. The reader is referred to [11] and [6] for surveys and further references.

2. Preliminaries

In this section we collect the definitions and notation that will be used throughout the article. The finite field of order q will be denoted by $\mathbb{F}_q$. The projective space associated with a vector space U will be denoted by $\mathrm{PG}(U)$. The $(r-1)$-dimensional projective space over the field $\mathbb{F}$ will be denoted by $\mathrm{PG}(\mathbb{F}^r)$ or $\mathrm{PG}(r-1,q)$ in case $\mathbb{F} = \mathbb{F}_q$. The sets of points, lines and hyperplanes of a projective space π will be denoted by $\mathcal{P}(\pi)$, $\mathcal{L}(\pi)$ and $\mathcal{H}(\pi)$, respectively; but we will often write π instead of $\mathcal{P}(\pi)$ when the meaning is clear. A *subgeometry* of a projective space $\mathrm{PG}(\mathbb{F}^r)$ is the set S of points for which there exists a frame with respect to which the homogeneous coordinates of points in S take values from a subfield $\mathbb{F}_0$ of $\mathbb{F}$, together with the subspaces generated by these points over $\mathbb{F}_0$. A subgeometry π_0 of $\mathrm{PG}(\mathbb{F}^r)$ is then isomorphic to $\mathrm{PG}(\mathbb{F}_0^r)$. If $\mathbb{F}_0$ has