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On $(0, \alpha)$-sets of generalized quadrangles

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Several infinite families of $(0, \alpha)$-sets, $\alpha \geq 1$, of finite classical and non-classical generalized quadrangles are constructed. When $\alpha = 1$ a $(0, \alpha)$-set of a generalized quadrangle is a partial ovoid. We construct a maximal partial ovoid of $\mathcal{H}(4, q^2)$, for any q, of size $2q^3 + q^2 + 1$, which generalizes the unique largest partial ovoid of $\mathcal{H}(4, 4)$ of size 21 found in [11], and a maximal partial ovoid of $\mathcal{Q}^-(5, q)$ of size $(q + 1)^2$, for any q. A tight set of a $GQ(q - 1, q + 1)$ is also provided.

1. Introduction

A *generalized quadrangle* of order (s, t) ($GQ(s, t)$ for short) is an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ of points and lines with the properties that any two points (lines) are incident with at most one line (point), every point is incident with $t + 1$ lines, every line

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is incident with $s+1$ points, and for any point P and line l which are not incident, there is a unique point on l collinear with P. The standard reference is [30]. A polar space of rank 2 is a GQ and a classical GQ is a GQ arising from a polar space. Also, the dual of a $GQ(s,t)$ is a $GQ(t,s)$.

A $(0,\alpha)$-set of a generalized quadrangle $\mathcal{S}$ is a non-empty set of points of $\mathcal{S}$ which intersects every line of $\mathcal{S}$ in either 0 or α points.

When α equals 2, then a $(0,2)$-set is called a *hyperoval* or a *BLT-set* of points [32, Section 6]. As observed in [8, Remark 2, p. 404], a hyperoval of a generalized quadrangle $\mathcal{S}$ is a regular graph of degree equal to $|\mathrm{Cone}_P(\mathcal{S})|$, valency $t+1$, and has the remarkable property of being triangle free.

A set of points $\mathcal{T}$ of a $GQ(s,t)$ is an *i-tight set* if for every point P in $\mathcal{T}$, there are $s+i$ points of $\mathcal{T}$ collinear with P, and for every point P not in $\mathcal{T}$, there are i points of $\mathcal{T}$ collinear with P. The size of an i-tight set of a $GQ(s,t)$ is $i(s+1)$.

Lower and upper bounds on the size of a $(0,\alpha)$-set of a generalized quadrangle of order (s,t) were obtained by Cameron, Hughes, Pasini [9] and Del Fra, Ghinelli, Payne [21]. For hyperovals in finite classical polar spaces see [19].

Proposition 1.1. *(See [9].) Let $\mathcal{S}$ be a generalized quadrangle of order (s,t) and let $\mathcal{K}$ be a $(0,\alpha)$-set of $\mathcal{S}$, with $k=|\mathcal{K}|$. Then*

i) *α is a divisor of k;*
ii) *$k \geq \alpha[(\alpha-1)t+1] = b_1$ and, if $\alpha \neq 1$, equality holds if and only if $\mathcal{K}$ is a subquadrangle of $\mathcal{S}$ of order $(\alpha-1,t)$;*
iii) *$k \geq (s+1)[\alpha(t+1)-(s+t)] = b_2$ and equality holds if and only if $\mathcal{K}$ is a $k/(s+1)$-tight set of $\mathcal{S}$ (in this case α divides $s+t$);*
iv) *$k \leq \alpha(st+1)$ and equality holds if and only if $\mathcal{K}$ is an α-ovoid of $\mathcal{S}$.*

For information on $(0,\alpha)$-sets of polar spaces see [21] and literature therein [19]. For more details on m-ovoids and tight sets of polar spaces, see [5,6].

Remark 1.2. Notice that b_1 is a better lower bound than b_2 if and only if $\alpha < 1+s/t$ and if $\alpha = 1+s/t$, then $b_1 = b_2$.

A *partial ovoid* ($(0,1)$-set) $\mathcal{O}$ of a generalized quadrangle $\mathcal{S}$ is a set of points of $\mathcal{S}$ such that every line contains at most one point of $\mathcal{O}$. A partial spread $\mathcal{F}$ of a generalized quadrangle $\mathcal{S}$ is a set of pairwise disjoint lines of $\mathcal{S}$. A partial ovoid or a partial spread is said to be *maximal* if it is maximal with respect to set-theoretic inclusion.

An *ovoid* $\mathcal{O}$ of a generalized quadrangle $\mathcal{S}$ is a set of points of $\mathcal{S}$ such that every line contains exactly one point of $\mathcal{O}$. A *spread* $\mathcal{F}$ of a generalized quadrangle $\mathcal{S}$ is a set of lines of $\mathcal{S}$ partitioning the point set of $\mathcal{S}$. For $\mathcal{W}(3,q)$ (q odd), $\mathcal{Q}^-(5,q)$ and $\mathcal{H}(4,q^2)$, it is known that no ovoids exist [33]. In this paper we provide some new infinite families of $(0,\alpha)$-set on certain classical and non-classical generalized quadrangles. Specializing to