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p-adic Gamma function and the polynomials $x^d + ax + b$ and $x^d + ax^{d-1} + b$ over $\mathbb{F}_q$

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ABSTRACT

In [14], McCarthy defined a function ${}_nG_n[\cdots]$ using the Teichmüller character of finite fields and quotients of the p-adic gamma function. He expressed the trace of Frobenius of elliptic curves in terms of special values of ${}_2G_2[\cdots]$. For $d \geqslant 2$, we establish four different expressions for the number of distinct zeros of the polynomials $x^d + ax + b$ and $x^d + ax^{d-1} + b$ over $\mathbb{F}_q$ in terms of special values of the function ${}_{d-1}G_{d-1}[\cdots]$.

1. Introduction and statement of results

Let p be an odd prime, and let $\mathbb{F}_q$ denote the finite field with q elements, where $q = p^r$, $r \geqslant 1$. Let $\mathbb{Z}_p$ denote the ring of p-adic integers, $\mathbb{Q}_p$ the field of p-adic numbers, $\overline{\mathbb{Q}_p}$ the algebraic closure of $\mathbb{Q}_p$, and $\mathbb{C}_p$ the completion of $\overline{\mathbb{Q}_p}$. Let $\Gamma_p(.)$ denote the Morita's

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p-adic gamma function, and let ω denote the Teichmüller character of $\mathbb{F}_q$. We denote the inverse of ω by $\overline{\omega}$. For $x \in \mathbb{Q}$ we let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x and $\langle x \rangle$ denote the fractional part of x, i.e. $x - \lfloor x \rfloor$. In [14], McCarthy defined a function ${}_nG_n[\cdots]$ as given below.

Definition 1.1. (See [14, Defn. 5.1].) Let $q = p^r$, for p an odd prime and $r \in \mathbb{Z}^+$, and let $t \in \mathbb{F}_q$. For $n \in \mathbb{Z}^+$ and $1 \leqslant i \leqslant n$, let $a_i, b_i \in \mathbb{Q} \cap \mathbb{Z}_p$. Then the function ${}_nG_n[\cdots]$ is defined by

$$\begin{aligned}
&{}_nG_n\left[\begin{matrix} a_1, & a_2, & \dots, & a_n \\ b_1, & b_2, & \dots, & b_n \end{matrix} \Big| t \right]_q \\
&:= \frac{-1}{q-1}\sum_{j=0}^{q-2}(-1)^{jn}\overline{\omega}^j(t) \\
&\quad\times \prod_{i=1}^{n}\prod_{k=0}^{r-1}(-p)^{-\lfloor \langle a_i p^k\rangle - \frac{jp^k}{q-1}\rfloor - \lfloor \langle -b_i p^k\rangle + \frac{jp^k}{q-1}\rfloor} \\
&\quad\times \frac{\Gamma_p(\langle (a_i - \frac{j}{q-1})p^k\rangle)}{\Gamma_p(\langle a_i p^k\rangle)} \frac{\Gamma_p(\langle (-b_i + \frac{j}{q-1})p^k\rangle)}{\Gamma_p(\langle -b_i p^k\rangle)}.
\end{aligned}$$

In [8], Greene introduced the notion of hypergeometric functions over finite fields. Mathematicians have established many connections between hypergeometric functions and number of zeros of polynomials. But these results are restricted to primes satisfying certain congruence conditions. For example, see [1,2,7,12,13].

In [14], McCarthy expressed the trace of Frobenius of elliptic curves in terms of special values of the function ${}_2G_2[\cdots]$ without any congruence condition on p. This G-function generalizes Greene's finite field hypergeometric function to the p-adic setting. For more details, see [14] and for an earlier version of this G-function, see [15].

In [4], the authors have found two different expressions for the trace of Frobenius of elliptic curves over $\mathbb{F}_q$ in terms of special values of the function ${}_2G_2[\cdots]$ with different parameters. It would be interesting to obtain similar results involving special values of the function ${}_nG_n[\cdots]$ for $n \geqslant 3$. In this paper we consider this problem.

Recently, in [3], the first author and Kalita gave two formulas for the number of distinct zeros of the polynomial $x^d + ax + b$ defined over $\mathbb{F}_q$ in terms of special values of ${}_dF_{d-1}$ and ${}_{d-1}F_{d-2}$ Gaussian hypergeometric series with characters of orders d and $d-1$ as parameters under the condition that $q \equiv 1 \pmod{d(d-1)}$. In [5], the authors obtained similar formulas for the number of zeros of more general polynomials $x^d + ax^i + b$ and $x^d + ax^{d-i} + b$ over $\mathbb{F}_q$. In this paper, for $d \geqslant 2$, we consider the polynomials $P_d(x) := x^d + ax + b$ and $Q_d(x) := x^d + ax^{d-1} + b$ over $\mathbb{F}_q$; and express their number of distinct zeros in terms of special values of the function ${}_{d-1}G_{d-1}[\cdots]$ without any congruence condition on q. We prove the following main results.